

8.324 Relativistic Quantum Field Theory II

2.4: UNSTABLE PARTICLES AND RESONANCES

We recall from previous lectures that the Feynman propagator for a general interacting theory is given by

$$G_F(p^2) = \Gamma(s = -p^2 + i\epsilon), \tag{1}$$

where

$$\Gamma(s) = \sum_j \frac{iZ_j}{s - m_j^2} + \int_{4m_1^2}^{\infty} d\mu^2 \sigma(\mu^2) \frac{i}{s - \mu^2}, \tag{2}$$

and the spectral density function is given by

$$\begin{aligned} \rho(\mu^2) &= \frac{1}{\pi} \Im \mathfrak{m}(iG_F(p^2)|_{p^2=-\mu^2}) \\ &= \sum_j Z_j \delta(\mu^2 - m_j^2) + \sigma(\mu^2). \end{aligned}$$

For $\mu^2 > 4m_1^2$, we have that $\rho(\mu^2) = \sigma(\mu^2) = \frac{1}{2\pi} \text{Disc}(\Gamma(s)|_{s=-\mu^2})$. The reason for this is that we cannot have

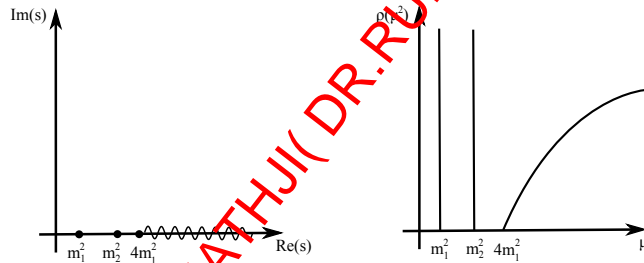


Figure 1: Plots of the function $\Gamma(s)$ on the complex plane and the spectral density function $\rho(\mu^2)$.

$m_j^2 > 4m_1^2$ for any particle j which interacts with the others. We can imagine changing the parameters of the theory, so that the pole moves beyond the branch point and sits on the branch cut. We will consider what happens in this case. Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}m_\chi^2\chi^2 + \frac{1}{2}g_\phi\phi^3 + \dots, \tag{3}$$

and examine the Feynman propagator $G_F^\phi(p^2)$. Suppose that $2m_\chi > m_\phi > m_\chi$. Then, perturbatively,

$$iG_F^\phi(p^2) = \frac{1}{p^2 + m_\phi^2 - \Pi^{(\phi)}(p^2) - i\epsilon}. \tag{4}$$

The three-point vertex rule is given by

$$\begin{array}{c} \phi \\ \diagdown \\ \text{---} \\ \diagup \\ \phi \end{array} \text{---} \chi \text{---} = ig, \tag{5}$$

and hence, to lowest order,

$$\Pi^{(\phi)}(p^2) = \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \tag{6}$$

and we see that $\Pi^{(\phi)}(p^2)$ becomes imaginary if $-p^2 > 4m_\chi^2$. Naively, the pole just sits on the branch cut. But, as we will show, the pole, in fact, moves to the other sheet of the complex plane. Physically, the particle becomes unstable since it can decay $\phi \rightarrow 2\chi$, where the imaginary part of the pole is proportional to the decay rate. In the Hilbert space of physical states, no states correspond to a single ϕ -particle. It becomes a resonance.

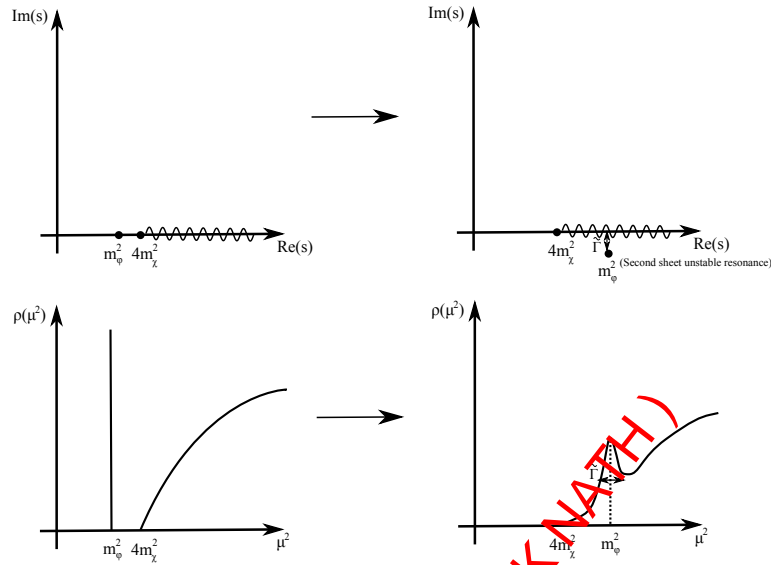


Figure 2: Formation of an unstable resonance for $m_\phi > 2m_\chi$.

From the last lecture, after subtracting divergences using counter-terms,

$$\Pi^{(\phi)}(p^2) = \frac{\alpha}{2} \int_0^1 dx D \log D - a' p^2 - b' m_\phi^2, \quad (7)$$

where $\alpha \equiv \frac{g^2}{(4\pi)^3}$, $D = m_\chi^2 + x(1-x)p^2$, and a' and b' are finite real constants, determined by the physical mass and physical field conditions:

$$\begin{aligned} \Pi(p^2 = -m^2) &= 0, \\ \frac{d\Pi}{dp^2}(p^2 = -m^2) &= 0. \end{aligned}$$

But, for $m_\phi^2 > 4m_\chi^2$, $\log D$ is complex at $p^2 = -m_\phi^2$, and there is no way to satisfy these two conditions for real a' and b' . Therefore, $G_F^{(\phi)}(p^2)$ cannot have a pole on the real axis. Physically, ϕ cannot be a physical single-particle state in \mathcal{H} . What can be done to resolve this is to modify the physical mass and physical field conditions to:

$$\begin{aligned} \Re \left(\Pi(p^2 = -m^2) \right) &= 0, \\ \Re \left(\frac{d\Pi}{dp^2}(p^2 = -m^2) \right) &= 0. \end{aligned}$$

which gives suitable real solutions for a' and b' . We then have

$$\Pi(p^2) = \frac{\alpha}{2} \int_0^1 dx D \log \frac{D}{|D_0|} - \frac{\alpha}{12} (p^2 + m_\phi^2) \quad (8)$$

with $D_0 = D(p^2 = -m_\phi^2) = m_\chi^2 - x(1-x)m_\phi^2$. Let us define $m_\phi \tilde{\Gamma} \equiv \Im \Pi(p^2)|_{p^2=-m_\phi^2}$. Then, near $p^2 = -m_\phi^2$, we have

$$\begin{aligned} iG_F(p^2) &= \frac{1}{p^2 + m_\phi^2 - \Pi(p^2) - i\epsilon} \\ &= \frac{1}{p^2 + m_\phi^2 - im_\phi \tilde{\Gamma}}. \end{aligned}$$

Thus, there is a pole at $p^2 = -m_\phi^2 + im_\phi \tilde{\Gamma}$. In terms of the complex function $\Gamma(s)$, there is a pole at

$$s = m_\phi^2 - im_\phi \tilde{\Gamma}, \quad (9)$$

and so, around the ϕ -mass, the spectral density function takes the form

$$\rho(\mu^2)|_{\mu^2 \approx m_\phi^2} = \frac{1}{\pi} \Im(iG_F(p^2)) = \frac{1}{\pi} \frac{m_\phi \tilde{\Gamma}}{(\mu^2 - m_\phi^2)^2 + m_\phi^2 \tilde{\Gamma}^2} + \dots \quad (10)$$

Now we proceed to calculate $\tilde{\Gamma}$:

$$\Im(\Pi(p^2)|_{p^2=-m_\phi^2}) = \frac{\alpha}{2} \int_0^1 dx \Im(D_0 \log \frac{D_0}{|D_0|}), \quad (11)$$

where $D_0 = m_\chi^2 - x(1-x)m_\phi^2 - i\epsilon$. Now, $D_0 < 0$ for the range

$$\frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4m_\chi^2}{m_\phi^2}} < x < \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4m_\chi^2}{m_\phi^2}}, \quad (12)$$

and in this region, $\log \frac{D_0}{|D_0|} = -i\pi$. Therefore, letting $\beta = \sqrt{1 - \frac{4m_\chi^2}{m_\phi^2}}$, we have

$$\begin{aligned} \Im(\Pi(-m_\phi^2)) &= -\frac{\alpha\pi}{2} \int_{\frac{1}{2}(1-\beta)}^{\frac{1}{2}(1+\beta)} dx (m_\chi^2 - x(1-x)m_\phi^2) \\ &= \frac{\pi\alpha}{12} m_\phi^2 \beta^3, \end{aligned}$$

and hence,

$$\tilde{\Gamma} = \frac{\pi\alpha}{12} m_\phi \beta^3. \quad (13)$$

In the problem set, we will see that when $m_\phi > 2m_\chi$, we have

$$\tilde{\Gamma}_{\phi \rightarrow 2\chi} = \frac{\pi\alpha}{12} m_\phi \left(1 - \frac{4m_\chi^2}{m_\phi^2}\right)^{\frac{3}{2}}, \quad (14)$$

and so $\tilde{\Gamma} = \tilde{\Gamma}_{\phi \rightarrow 2\chi}$. We now consider a couple of additional points.

Remarks:

1. Consider the function $\log z$. The function increases by 2π as we wrap around the origin each time. This introduces a discontinuity if we try to plot the function on one sheet of the complex plane. Instead, we have to plot the function on multiple sheets, introducing a branch cut, for example on the first sheet at $\Im(\log(-x + i\epsilon)) = i\pi$, $\Im(\log(-x - i\epsilon)) = -i\pi$. Similarly, with $\Gamma(s)$, there is a branch cut at $x = 4m_\chi^2$. When we calculated

$$G_F(p^2) = \frac{1}{p^2 + m_\phi^2 - im_\phi \tilde{\Gamma}} \quad (15)$$

we treated it as an analytic function across the real axis. In other words, we have done the analytic continuation below the cut, and so, the pole lies on the second sheet.

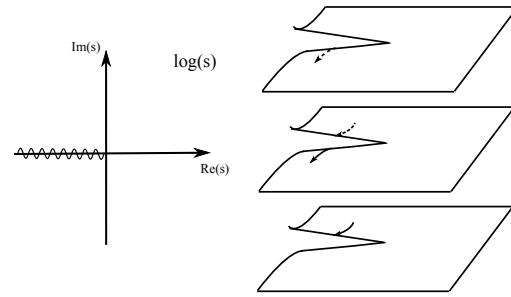


Figure 3: The branch cut in $\log(s)$ and the additional sheets needed to plot the function smoothly.

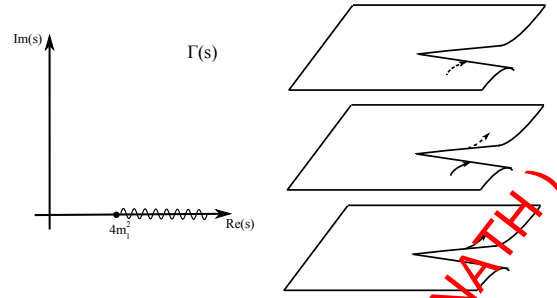


Figure 4: The branch cut in $\Gamma(s)$ with the second sheet on which the pole lies.

- One can show that, as a consequence of the optical theorem,

$$\begin{aligned}
 \Im(\Pi(p^2)) &= \text{Im} \left(\text{---} \text{---} \text{---} \right) \\
 &= \left| \text{---} \text{---} \right|^2,
 \end{aligned}
 \tag{16}$$

(Note: The diagrams above are Feynman diagrams. The first is a loop diagram with a dashed line and a solid line. The second is a tree-level diagram with a solid line and a dashed line.)

where the propagators are put on shell, replaced by $2\pi\delta(p_1^2 + m_\chi^2)\Theta(p_1^0)$. This will be discussed in the problem set.

- Considering the propagation of an unstable particle, for $t > 0$,

$$G_F(t, \vec{p}) = \langle \phi(t, \vec{p})\phi(0, \vec{p}) \rangle = \int \frac{dw}{2\pi} e^{-i\omega t} \frac{1}{p^2 + m^2 - im\tilde{\Gamma}},
 \tag{17}$$

with $p^2 = -\omega^2 + \vec{p}^2$. Hence,

$$G_F(t, \vec{p}) = \frac{e^{-i\sqrt{\omega_{\vec{p}}^2 - im\tilde{\Gamma}}t}}{\sqrt{\omega_{\vec{p}}^2 - im\tilde{\Gamma}}}
 \tag{18}$$

with $\omega_{\vec{p}} \equiv \sqrt{\vec{p}^2 + m^2}$. Now, for the case that $\tilde{\Gamma} \ll \frac{\omega_{\vec{p}}^2}{m}$, this reduces to

$$\begin{aligned}
 G_F(t, \vec{p}) &= \frac{1}{2\omega_{\vec{p}}} e^{-i\omega_{\vec{p}}t - \frac{m\tilde{\Gamma}}{2\omega_{\vec{p}}}t} \\
 &= \frac{1}{2\omega_{\vec{p}}} e^{-i\omega_{\vec{p}}t - \frac{t}{2\tau(\vec{p})}}
 \end{aligned}$$

where we have defined the lifetime $\tau(\vec{p}) \equiv \frac{\omega_{\vec{p}}}{m}\tilde{\Gamma}^{-1} = \frac{1}{\sqrt{1-v^2}}\tilde{\Gamma}^{-1}$, which we recognise as the effect of time dilation, where $\tilde{\Gamma}^{-1}$ is the lifetime of the particle in its rest frame.