8.324 Relativistic Quantum Field Theory II

Lecture 11

2.5: S-MATRIX ELEMENTS AND LSZ REDUCTION

Having studied in detail the general structure of two-point functions, let us now look at the general structure of higher-point functions. These are of course much more complicated, and the power of Lorentz and translational symmetries becomes much more limited. Nevertheless, there are some important statements to be made. We again consider a scalar field for illustration. Generalizations to spinors, vectors and multiple fields do not contain additional conceptual insights.

We consider

$$G_F(x_1, \ldots, x_n) \equiv \langle 0| T(\phi(x_1)\ldots\phi(x_n)) |0\rangle , \tag{1}$$

where $|0\rangle$ is the exact vacuum of the interacting state, and the Fourier transform is given by

$$G_F(p_1, \ldots, p_n) \equiv \int d^4x_1 \ldots d^4x_n \, e^{-i(p_1.x_1+\ldots+p_n.x_n)} G_F(x_1, \ldots, x_n), \tag{2}$$

where $p_i^\mu = (\omega_i, \vec{p}_i)$. Now, by translational invariance, we have that $G_F(x_1, \ldots, x_n) = G(0, x_2 - x_1, \ldots, x_n - x_1)$, and so

$$G_F(p_1, \ldots, p_n) \propto (2\pi)^4 \delta^{(4)}(p_1 + \ldots + p_n). \tag{3}$$

Now, if we consider any combination of these momenta, for example

$$p \equiv p_1 + p_2 + \ldots + p_r = -(p_{r+1} + \ldots + p_n), \quad 1 \leq r \leq n-1, \tag{4}$$

then $G_F(p_1, \ldots, p_n)$ has a pole at $p^2 = -m_i^2$, where m_i is the mass of a single-particle state.

Proof: Let us first consider $r = 1$, that is, consider

$$G_F(p, y_1, \ldots, y_{n-1}) = \int d^4x \, e^{-ip.x} \langle 0| T(\phi(x)\phi(y_1)\ldots\phi(y_{n-1})) |0\rangle . \tag{5}$$

The integration over $x^0 = t$ can be separated into three regions:

$$\begin{aligned} \int dt &= \int_{\mathbf{I}} dt + \int_{\mathbf{II}} dt + \int_{\mathbf{III}} dt, \\ &= \int_{T_+}^{\infty} dt + \int_{T_-}^{T_+} dt + \int_{-\infty}^{T_-} dt \end{aligned}$$

where $T_- < y_1^0, \ldots y_{n-1}^0 < T_+$. In region **I**,

$$G_F(x, y_1, \ldots y_{n-1}) = \langle 0| \phi(x) T(\phi(y_1)\ldots\phi(y_{n-1})) |0\rangle . \tag{6}$$

We can now use the same trick as in the case of the two-point function, inserting a complete set of physical states:

$$1 = \sum_n |n\rangle \langle n| = \sum_j \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}^{(j)}} \left|j, \vec{k}\right\rangle \left\langle j, \vec{k}\right| + \text{multi-particle states}, \tag{7}$$

where $\omega_{\vec{k}}^{(j)} = \sqrt{\vec{k}^2 + m_j^2}$. For simplicity, let us consider a single species. Then, we have that

$$\begin{aligned} G_F(x, y_1, \ldots, y_{n-1}) &= \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} \left\langle 0 \left|\phi(x)\right| \vec{k}\right\rangle \left\langle \vec{k} \left|T(\phi(y_1)\ldots\phi(y_{n-1}))\right| 0\right\rangle \\ &= \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} e^{ik.x} \sqrt{Z} \left\langle \vec{k} \left|T(\phi(y_1)\ldots\phi(y_{n-1}))\right| 0\right\rangle , \end{aligned}$$

and so, for the integral over region I, we have

$$\begin{aligned}\mathbf{I} &= \sqrt{Z}\int_{T_+}^{\infty} dt \int d\vec{x} e^{i\omega t - i\vec{p}.\vec{x}} \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}\left\langle \vec{k}\,|T(\ldots)|\,0\right\rangle \\ &= \sqrt{Z}\frac{1}{2\omega_{\vec{p}}}\frac{ie^{i(\omega-\omega_{\vec{p}})T_+}}{\omega-\omega_{\vec{p}}+i\epsilon}\langle\vec{p}|\,T(\ldots)\,|0\rangle \\ &\xrightarrow[\omega\to\omega_{\vec{p}}]{} \sqrt{Z}\frac{-i}{p^2+m^2-i\epsilon}\langle\vec{p}|\,T(\ldots)\,|0\rangle\,.\end{aligned}$$

Similarly, for the integral over region **III**, we find

$$\mathbf{III} \xrightarrow[\omega\to-\omega_{\vec{p}}]{} \sqrt{Z}\frac{-i}{p^2+m^2-i\epsilon}\langle 0|\,T(\ldots)\,|{-\vec{p}}\rangle\,. \tag{8}$$

Region **II** is a compact integral, so it does not have singular behaviour. So we conclude that, as a function of p,

$$\begin{aligned}G_F(p,\ldots) &\xrightarrow[\omega\to\omega_{\vec{p}}]{} \frac{-i\sqrt{Z}}{p^2+m^2-i\epsilon}\langle\vec{p}|\,T(\ldots)\,|0\rangle\,, \\ &\xrightarrow[\omega\to-\omega_{\vec{p}}]{} \frac{-i\sqrt{Z}}{p^2+m^2-i\epsilon}\langle 0|\,T(\ldots)\,|{-\vec{p}}\rangle\,.\end{aligned}$$

The above argument can be generalized to any r. Consider $p = p_1 + \ldots + p_r = -(p_{r+1} + \ldots + p_n)$. Among all possible orderings of $t_1, \ldots, t_n$, we consider those with $\min\{t_1, \ldots, t_r\} > \max\{t_{r+1}, \ldots, t_n\}$. In this case, we have

$$G_F(x_1,\ldots,x_n) = \Theta(\tau)\langle 0|\,T(\phi(x_1)\ldots\phi(x_r))T(\phi(x_{r+1})\ldots\phi(x_n))\,|0\rangle + \text{other orderings}, \tag{9}$$

with $\tau = \min\{t_1, \ldots, t_r\} - \max\{t_{r+1}, \ldots, t_n\}$. Hence, inserting a complete set of intermediate states, including one-particle states, we have

$$G_F(x_1,\ldots,x_n) = \Theta(\tau)\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}\left\langle 0\,|T(\phi(x_1)\ldots\phi(x_r)|\,\vec{k}\right\rangle\left\langle\vec{k}\,|T(\phi(x_{r+1})\ldots\phi(x_n))|\,0\right\rangle, \tag{10}$$

and we proceed to take the Fourier transform $G_F(x_1, \ldots, x_n) \longrightarrow G_F(p_1, \ldots, p_n)$. The analysis of the Fourier transform is a bit more intricate than the $r = 1$ case. The details are left to the reader; they can be found in *Weinberg, Volume I*, §10.2. The result is that

$$G_F(p_1,\ldots,p_n) \xrightarrow[p^0\to E_{\vec{p}}]{} (2\pi)^4\delta^{(4)}(p_1+\ldots+p_n)\frac{-i}{p^2+m^2-i\epsilon}M_{0|\vec{p}}(p_2,\ldots,p_r)M_{\vec{p}|0}(p_{r+2},\ldots,p_n) \tag{11}$$

where

$$M_{0|\vec{p}}(p_2,\ldots,p_r) = \int d^4y_2\ldots d^4y_r\,e^{-ip_2\cdot y_2-\ldots-ip_r\cdot y_r}\langle 0|\,T(\phi(0)\phi(y_2)\ldots\phi(y_r))\,|\vec{p}\rangle\,, \tag{12}$$

and

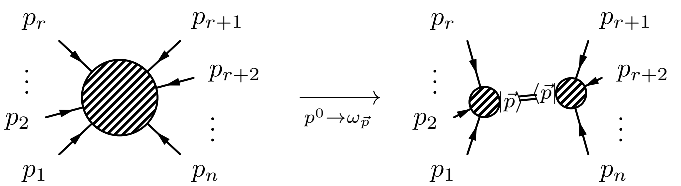
$$M_{\vec{p}|0}(p_{r+2},\ldots,p_n) = \int d^4y_{r+2}\ldots d^4y_n\,e^{-ip_{r+2}\cdot y_{r+2}-\ldots-ip_n\cdot y_n}\langle\vec{p}|\,T(\phi(0)\phi(y_{r+2})\ldots\phi(y_n))\,|0\rangle\,. \tag{13}$$

□

Remarks:

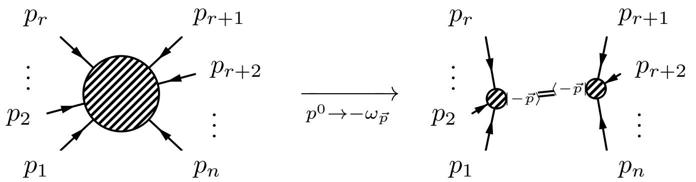
1. The result is generally valid for any interacting theory, and is non-perturbative in nature. In particular, as in the case of the two-point function, the single-particle states do not have to correspond to fields which appear in the Lagrangian. Further, ϕ does not have to be a fundamental field appearing in the Lagrangian: the same conclusion applies if one uses composite operators. For example, in quantum chromodynamics, pions can appear as such a pole.

2. The result is physically intuitive. Diagramatically, it can be expressed as, when $p = p_1 + \ldots + p_r$ is close to on-shell for a particle of mass m,



$$\quad (14)$$

where the internal propagator contributes a factor of $\frac{-i}{p^2+m^2-i\epsilon}$. Similarly,



$$\quad (15)$$

2.5.2: LSZ Reduction

Now, let us take p_1, p_2 to be simultaneously on-shell. That is, $p_1^0 \approx -\omega_{\vec{p}_1}$, $p_2^0 \approx -\omega_{\vec{p}_2}$. Then the Green's function is given by

$$G_F(p_1, p_2, \ldots, p_n) \longrightarrow \frac{-i\sqrt{Z_1}}{p^2+m_1^2-i\epsilon}\frac{-i\sqrt{Z_2}}{p^2+m_2^2-i\epsilon}\left\langle \vec{p}\right| T(\phi_3(x_3)\ldots\phi_n(x_n))\left|-\vec{p}_1, -\vec{p}_2\right\rangle. \quad (16)$$

If we now take $p_n^0 \approx \omega_{\vec{p}_3}, \ldots, p_n^0 \approx \omega_{\vec{p}_n}$, we find

$$G_F(p_1, p_2, \ldots, p_n) \longrightarrow \prod_{j=1}^{n} \frac{-i\sqrt{Z_j}}{p^2+m_j^2-i\epsilon}\left\langle \vec{p}_3, \ldots, \vec{p}_n \right| -\vec{p}_1, -\vec{p}_2\rangle, \quad (17)$$

giving the $S-$matrix element $\left\langle \vec{p}_3, \ldots, \vec{p}_n \right| -\vec{p}_1, -\vec{p}_2\rangle$. This suggests that we can calculate the S-matrix elements using Feynman diagrams as follows:

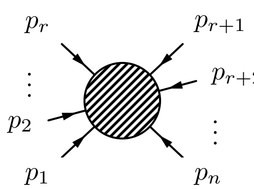
1. Consider all Feynman diagrams for the Green's function $G(p_1, \ldots, p_n)$.

2. Put all the external momenta on shell:

$$\begin{cases} p_i^0 \to -\omega_{\vec{p}_i} & \text{for initial momenta,} \\ p_f^0 \to \omega_{\vec{p}_f} & \text{for final momenta.} \end{cases} \quad (18)$$

3. Obtain amptutated amplitudes by discarding the external propagators.

4. Since an external propagator behaves as $\frac{-i\sqrt{Z}}{p^2+m^2-i\epsilon}$ near the mass-shell, we have

$$\langle f|\, i\rangle = \prod_{j=1}^{n}\sqrt{Z_j}$$  $$\quad (19)$$

where the shaded vertex denotes all amputated diagrams.

2.6: THE OPTICAL THEOREM

The optical theorem is a simple consequence of the unitarity of the S-matrix. Since

$$S^\dagger S = 1, \quad (20)$$

where it is convential to write $S = 1 + iT$, we have that

$$-i(T - T^\dagger) = T^\dagger T. \quad (21)$$

We now take the matrix elements of this equation between some states

$$-i\left\langle b\right|(T - T^\dagger)\left|a\right\rangle = \left\langle b\right| T^\dagger T\left|a\right\rangle \quad (22)$$

where we have $\langle b| T |a\rangle \equiv M(a \to b)$, and so $\langle b| T^\dagger |a\rangle = M(b \to a)*$. Inserting a complete set of states into the right-hand side of this equation, we obtain

$$\begin{aligned}\langle b| T^\dagger T |a\rangle &= \sum_n \langle b| T^\dagger |n\rangle \langle n| T |a\rangle \\ &= \sum_n M(a \to n) M(b \to n)*,\end{aligned}$$

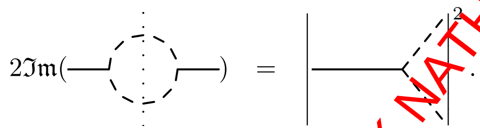
and so we obtain the result

$$-i\left[M(a \to b) - M(b \to a)*\right] = \sum_n |M(a \to n)|^2 \,. \tag{23}$$

In particular, if we take $a = b$, we find

$$2\mathfrak{Im}(M(a \to a)) = \sum_n |M(a \to n)|^2 \,, \tag{24}$$

relating the imaginary part of forward scattering to the total cross section. This result can also be applied order-by-order in perturbation theory. Recall the example we discussed in lecture 10: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