8.324 Relativistic Quantum Field Theory II

We now consider the Lagrangian for quantum electrodynamics in terms of renormalized quantities.

$$\begin{aligned}\mathscr{L} &= -\frac{1}{4}F^B_{\mu\nu}F^{\mu\nu}_B - i\bar{\psi}_B(\gamma^\mu(\partial_\mu - ie_B A^B_\mu) - m_B)\psi_B \\ &= -\frac{1}{4}Z_3 F_{\mu\nu}F^{\mu\nu} - iZ_2\bar{\psi}(\gamma^\mu\partial_\mu - m - \delta m)\psi - Z_2 e A_\mu \bar{\psi}\gamma^\mu\psi.\end{aligned}$$

We know from previous lectures that there is no mass term for A_μ, that the bare and physical fields and couplings are related by

$$\begin{aligned}A^B_\mu &= \sqrt{Z_3}A_\mu, \\ \psi_B &= \sqrt{Z_2}\psi, \\ m_B &= m + \delta m, \\ e_B &= \frac{1}{\sqrt{Z_3}}e,\end{aligned}$$

and that there is no renormalization for the gauge fixing term. These results are a consequence of gauge symmetry, enforced through the Ward identities. We split the Lagrangian into three pieces:

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1 + \mathscr{L}_{ct}, \tag{1}$$

where we have

$$\begin{aligned}\mathscr{L}_0 &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\bar{\psi}(\gamma^\mu\partial_\mu - m)\psi, \\ \mathscr{L}_1 &= -eA_\mu\bar{\psi}\gamma^\mu\gamma, \\ \mathscr{L}_{ct} &= -\frac{1}{4}(Z_3 - 1)F_{\mu\nu}F^{\mu\nu} - i(Z_2 - 1)\bar{\psi}(\gamma^\mu\partial_\mu - m)\psi \\ &\quad + iZ_2\delta m\bar{\psi}\psi - (Z_2 - 1)eA_\mu\bar{\psi}\gamma^\mu\gamma.\end{aligned}$$

$\mathscr{L}_0$ is the free Lagrangian, $\mathscr{L}_1$ is the interaction Lagrangian, and $\mathscr{L}_{ct}$ is the counter-term Lagrangian. The parameters $Z_3 - 1$, $Z_2 - 1$ and δm are specified by the following renormalization conditions:

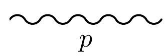
1. For the spinor propagator, $S(k) = \frac{1}{i\not{k} - m + i\epsilon - \Sigma(\not{k})}$,

$$\begin{aligned}\Sigma\big|_{\not{k}=-im} &= 0 \text{ (Physical mass condition)}, \\ \left.\frac{d\Sigma}{d\not{k}}\right|_{\not{k}=-im} &= 0 \text{ (Physical field condition)}.\end{aligned}$$

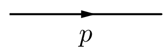
2. For the photon propagator, $D^T_{\mu\nu}(k) = \frac{P^T_{\mu\nu}}{k^2 - i\epsilon}\frac{1}{1 - \Pi(k^2)}$,

$$\Pi\big|_{k=0} = 0 \text{ (Physical mass condition)}. \tag{2}$$

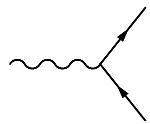
These three conditions allow us to fix our three parameters. We note that there is no need to introduce conditions on vertex corrections, and so, $\mathscr{L}$ is written in terms of physically measured masses and couplings. From this deconstruction, we acquire a set of Feynman rules for the interaction and counterterms in terms of the physical propagators.



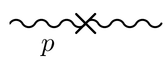
$$= \frac{-ig_{\mu\nu}}{k^2 + i\epsilon},$$



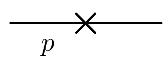
$$= \frac{1}{i\not{k} - m + i\epsilon},$$



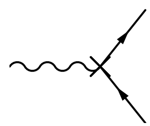
$$= -ie\gamma^\mu,$$



$$= -i(Z_3 - 1)(k^2 g^{\mu\nu} - k^\mu k^\nu) \sim O(e^2),$$



$$= -i(Z_2 - 1)(i\not{k} - m) + Z_2\delta m \sim O(e^2),$$

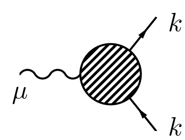


$$= -i(Z_2 - 1)e\gamma^\mu \sim O(e^3). \tag{3}$$

3.2: VERTEX FUNCTION

Consider the effective vertex we defined before:

$$\Gamma^\mu_{phys}(k, k) \quad =$$



$$\equiv \quad -ie_{phys}\gamma^\mu. \tag{4}$$

This is the physical vertex: it captures the full electromagnetic properties of a spinor interacting with a photon. As we showed in the previous lecture, the Ward identities impose that

$$\Gamma^\mu(k, k) = -ie\gamma^\mu \tag{5}$$

when k is on-shell, with $e = \frac{1}{\sqrt{Z_3}} e_B$ being the physical charge. We note that in this case, $q = 0$, and so this is an interaction with a static potential, measuring electric charge. We will now proceed to examine the general structure of $\Gamma^\mu(k_1, k_2)$, with k_1 and k_2 on-shell. We will discuss the physical interpretation, and we will compute the one-loop correction explicitly. For general $k_1^2 = k_2^2 = -m^2$, $q^2 = (k_2 - k_1)^2 \neq 0$, the process being described is an electron interacting a general external electromagnetic field. From Lorentz invariance, we can build Γ^μ from γ^μ, k_1^μ and k_2^μ. Hence,

$$i\Gamma^\mu(k_1, k_2) = \gamma^\mu A + i(k_2^\mu + k_1^\mu)B + (k_2^\mu - k_1^\mu)C, \tag{6}$$

where A, B, and C are 4×4 matrix functions of k_1 and k_2. But, since k_1 and k_2 are on-shell, and Γ^μ always appears in a product as

$$\bar{u}_{s'}(k_2)\Gamma^\mu(k_1, k_2)u_s(k_1) \tag{7}$$

where $u_{s'}(k_1)$ and $\bar{u}_s(k_2)$ are on-shell spinor wave functions, we can then simplify Γ^μ with the understanding that it will always be found in this combination, using the on-shell spinor identities

$$\begin{aligned} \not{k}u_s(k) &= -imu_s(k), \\ \bar{u}_s(k)\not{k} &= -im\bar{u}_s(k). \end{aligned}$$

Hence, A, B, and C are scalars, and functions of the scalars k_1^2, k_2^2 and $k_1.k_2$, or, equivalently, of q^2 and m. From the Ward identities, we have that

$$q_\mu \Gamma^\mu = 0, \tag{8}$$

and, as $\bar{u}_{s'}(k_2)\not{q}u_s(k_1) = 0$, and $q_\mu(k_1^\mu - k_2^\mu) = 0$, only the term in C on the left-hand side is non-zero. We therefore have $C = 0$. It is common to rewrite Γ^μ using the Gordon identity. Defining $\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$, this result states

$$\bar{u}_{s'}(k_2)\gamma^\mu u_s(k_1) = \frac{i}{2m}\bar{u}_{s'}(k_2)\left[(k_1^\mu + k_2^\mu) + iq_\nu\sigma^{\mu\nu}\right]u_s(k_1). \tag{9}$$

This allows us to exchange the term in B for a term in $\sigma^{\mu\nu}$.

Proof

$$\begin{aligned}
\bar{u}_{s'}(k_2)\gamma^\mu u_s(k_1) &= \frac{i}{2m}\left[\bar{u}_{s'}(k_2)\gamma^\mu \not{k}_1 u_s(k_1) + \bar{u}_{s'}(k_2)\not{k}_2\gamma^\mu u_s(k_1)\right] \\
&= \frac{i}{2m}\left[\left(\frac{k_{2v}+k_{1v}}{2} - \frac{k_{2v}-k_{1v}}{2}\right)\bar{u}_{s'}(k_2)\gamma^\mu\gamma^\nu u_s(k_1)\right. \\
&\quad \left. + \left(\frac{k_{2v}+k_{1v}}{2} + \frac{k_{2v}-k_{1v}}{2}\right)\bar{u}_{s'}(k_2)\gamma^\nu\gamma^\mu u_s(k_1)\right] \\
&= \frac{i}{2m}\bar{u}_{s'}(k_2)\left[\left(\frac{k_{2v}+k_{1v}}{2}\right)\{\gamma^\mu,\gamma^\nu\} - \left(\frac{k_{2v}-k_{1v}}{2}\right)[\gamma^\mu,\gamma^\nu]\right]u_s(k_1) \\
&= \frac{i}{2m}\bar{u}_{s'}(k_2)\left[(k_2^\mu + k_1^\mu) + iq_v\sigma^{\mu\nu}\right]u_s(k_1).
\end{aligned}$$

□

From this, we find that

$$i\Gamma^\mu(k_1,k_2) = e\left[\gamma^\mu F_1(q^2) - \frac{\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\right]. \tag{10}$$

$F_1(q^2)$ and $F_2(q^2)$ are known as form factors. We have that $eF_1(q^2) = A + 2mB$, and $eF_2(q^2) = -2mB$. Note that the Ward identity means that $F_1(0) = 1$ exactly.