# 8.324 Relativistic Quantum Field Theory II

## Lecture 15

### 3.3: ANOMALOUS MAGNETIC MOMENT

In the last lecture, we showed that the physical vertex $\Gamma^\mu(k_1, k_2)$ takes the general form

$$i\Gamma^\mu(k_1,k_2) = e\left[\gamma^\mu F_1(q^2) - \frac{\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\right], \tag{1}$$

and in the limit $k_1 - k_2 = q \longrightarrow 0$,

$$i\Gamma^\mu(k_1,k_2) \longrightarrow e\left[\gamma^\mu - \frac{\sigma^{\mu\nu}q_\nu}{2m}F_2(0)\right] \equiv \Gamma^\mu_{eff}(k_1,k_2). \tag{2}$$

This vertex is reproduced by the effective Lagrangian,

$$\mathscr{L}_{eff} = -i\bar{\psi}(\gamma^\mu\partial_\mu - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\gamma - \frac{ieF_2(0)}{4m}\bar{\psi}\sigma^{\mu\nu}F_{\mu\nu}\psi. \tag{3}$$

Consider the case where $\psi$ is non-relativistic, in a classical electromagnetic background $A_\mu$. That is,

$$p^0 \sim mv^2 + m \qquad \vec{p} \sim mv,$$
$$A^0 \sim mv^2 \qquad \vec{A} \sim mv,$$

where $v \ll 1$. This is consistent, because $D^\mu = \partial^\mu - ieA^\mu = i(p - eA)^\mu$, so, $A^0$ and $\vec{A}$ interact with $\psi$ and give it energy of the order $mv^2$, and momentum of the order $mv$. The Dirac equation now has the form

$$(\gamma^\mu(\partial_\mu - ieA_\mu) - m)\psi + \frac{eF_2(0)}{4m}F_{\mu\nu}\sigma^{\mu\nu}\psi = 0, \tag{4}$$

or $i\partial_t\psi = H\psi$, with

$$H = m\beta + \vec{\alpha}.(\vec{p} - e\vec{A}) + eA^0 + \frac{ieF_2(0)}{4m}\gamma^0\sigma^{\mu\nu}F_{\mu\nu}. \tag{5}$$

Here, $\beta = -i\gamma^0$ and $\alpha^i = -i\gamma^0\gamma^i$. We will choose the basis

$$\gamma^0 = i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = i\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{6}$$

or, equivalently,

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}. \tag{7}$$

From this, we find

$$\begin{aligned} \gamma^0\sigma^{0i} &= \frac{i}{2}\gamma^0\left[\gamma^0,\gamma^i\right] = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \\ \gamma^0\sigma^{ij} &= \frac{i}{2}\gamma^0\left[\gamma^i,\gamma^j\right] = -i\epsilon^{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix}. \end{aligned}$$

We now write $\psi^T \equiv \begin{pmatrix} \phi & \chi \end{pmatrix}$, $F_{0i} \equiv E_i$ and $F_{ij} = \epsilon_{ijk}B_k$. The Dirac equation reduces to two coupled partial differential equations:

$$\begin{aligned} i\partial_t\phi &= m\phi + \vec{\sigma}.(\vec{p} - e\vec{A})\chi + eA^0\phi + \frac{ieF_2(0)}{2m}\left[\vec{\sigma}.\vec{E}\chi - i\vec{\sigma}.\vec{B}\phi\right], \\ i\partial_t\chi &= -m\chi + \vec{\sigma}.(\vec{p} - e\vec{A})\phi + eA^0\chi + \frac{ieF_2(0)}{2m}\left[-\vec{\sigma}.\vec{E}\phi + i\vec{\sigma}.\vec{B}\chi\right]. \end{aligned}$$

We now let $\phi = e^{-imt}\Phi$, $\chi = e^{-imt}X$. As $i\partial_t\psi \sim \left[m + O(mv^2)\right]\psi$, $\Phi$ and $X$ describe fluctuations with $\Delta E \sim mv^2$. In terms of these fields, taking the limit $v \longrightarrow 0$, the equations reduce to

$$\begin{aligned} i\partial_t\Phi &= \vec{\sigma}.\vec{\pi}X + eA^0\Phi + \frac{ieF_2(0)}{2m}\left[-i\vec{\sigma}.\vec{B}\phi\right] + O(v^3), \\ 0 &= -2mX + \vec{\sigma}.\vec{\pi}\Phi + O(v^2), \end{aligned}$$

where $\vec{\pi} = \vec{p} - e\vec{A}$, and so, solving the second equation for $X$, and inserting the result into the first equation, we obtain

$$\begin{aligned} X &= \frac{1}{2m}\vec{\sigma}.\vec{\pi}\Phi, \\ i\partial_t\Phi &= \frac{1}{2m}(\vec{\sigma}.\vec{\pi})^2\Phi + eA^0\Phi + \frac{eF_2(0)}{2m}\vec{\sigma}.\vec{B}\Phi. \end{aligned}$$

Now,

$$(\vec{\sigma}.\vec{\pi})^2 = \sigma_i\sigma_j\pi^i\pi^j = (\delta_{ij} + i\epsilon_{ijk}\sigma_k)\pi^i\pi^j = \pi^2 + e\vec{\sigma}.\vec{B}, \tag{8}$$

as $\left[\pi^i, \pi^j\right] = -ieF^{ij} = -ie\epsilon_{ijk}B_k$, and so we arrive at the following time-evolution equation in the limit $v \longrightarrow 0$:

$$i\partial_t\Phi = \left[\frac{1}{2m}(\vec{p} - e\vec{A})^2 + eA^0 + \frac{e}{2m}(1 + F_2(0))\vec{\sigma}.\vec{B}\right]\Phi. \tag{9}$$



We recognise the first two terms as the kinetic energy of a particle in an electromagnetic field and the electrostatic potential energy, respectively, and the third term takes the form of a magnetic interaction,

$$H_{mag} = -\vec{\mu}.\vec{B}, \tag{10}$$

with

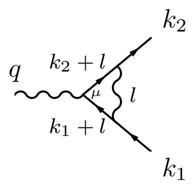
$$\vec{\mu} = -\frac{e}{2m}2(1 + F_2(0))\frac{\vec{\sigma}}{2} = \gamma\vec{S}, \tag{11}$$

with $\vec{S} = \frac{\vec{\sigma}}{2}$ the spin, and $\gamma = \frac{e}{2m}g$ the gyromagnetic ratio. Classically, we expect $g = 1$, and in the Dirac equation of quantum mechanics, we find $g = 2$. We see that in the case of quantum electrodynamics, we have $g = 2 + 2F_2(0)$. The additional term of $2F_2(0)$ is known as the anomalous magnetic moment. We will now explicitly compute the lowest order correction to the magnetic moment.

### 3.3.1: One-loop correction to the magnetic moment

To lowest order, the correction to the physical vertex function is given by

$$\Gamma_1^\mu(k_1, k_2) \equiv$$




$$= (-ie)^3\int\frac{d^4l}{(2\pi)^4}\gamma^\rho S_0(k_2 + l)\gamma^\mu S_0(k_1 + l)\gamma^\nu D^{(0)}_{\nu\rho}(l), \tag{12}$$

where $S_0(k) = \frac{-i\not{k} - m}{k^2 + m^2 - i\epsilon}$ and $D^{(0)}_{\mu\nu} = \frac{-ig_{\mu\nu}}{l^2 - i\epsilon}$. Explicitly, we have

$$\Gamma_1^\mu(k_1, k_2) = (-ie)^3(-1)^2(-i)\int\frac{d^4l}{(2\pi)^4}\frac{\gamma^\rho\left[i(\not{k}_2 + \not{l}) + m\right]\gamma^\mu\left[i(\not{k}_1 + \not{l}) + m\right]\gamma^\nu}{((k_1 + l)^2 + m^2 - i\epsilon)\,((k_2 + l)^2 + m^2 - i\epsilon)\,((l^2 - i\epsilon)}. \tag{13}$$

We can combine the denominators using the Feynman trick,

$$\frac{1}{A_1A_2A_3} = \int_0^1 dx_1\int_0^1 dx_2\int_0^1 dx_3\,\delta(x_1 + x_2 + x_3 - 1)\frac{2}{(x_1A_1 + x_2A_2 + x_3A_3)^3}, \tag{14}$$

reducing our result for $\Gamma_1^\mu$ to

$$\Gamma_1^\mu = 2e^3 \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \delta(x_1+x_2+x_3-1)\frac{N^\mu}{D^3}, \tag{15}$$

where

$$N^\mu \equiv \gamma^\nu \left[i(\not{k}_2 + \not{l}) + m\right]\gamma^\mu \left[i(\not{k}_1+\not{l})+m\right]\gamma_\nu, \tag{16}$$

and

$$\begin{aligned} D &\equiv x_1\left[(k_1+l)^2+m^2\right] + x_2\left[(k_2+l)^2+m^2\right] + x_3 l^2 - i\epsilon \\ &= (l+x_1k_1+x_2k_2)^2 + x_1(1-x_1)k_1^2 + x_2(1-x_2)k_2^2 - 2x_1x_2k_1.k_2 + (x_1+x_2)m^2 - i\epsilon. \end{aligned}$$

We may shift the variable in the integral $l \longrightarrow p = l + x_1k_1 + x_2k_2$, and rewrite the result in terms of $q^2$ instead of $k_1.k_2$, giving

$$D = p^2 + x_1x_2q^2 + (x_1+x_2)^2m^2 - i\epsilon. \tag{17}$$

Further,

$$\begin{aligned} N^\mu &= \gamma^\nu\left[i(\not{p} - x_1\not{k}_1 + (1-x_2)\not{k}_2) + m\right]\gamma^\mu\left[i(\not{p}+(1-x_1)\not{k}_1 - x_2\not{k}_2)+m\right]\gamma_\nu \\ &= -\gamma^\nu\not{p}\gamma^\mu\not{p}\gamma_\nu + \gamma^\nu\left[i(x_1\not{k}_1 + (1-x_2)\not{k}_2)+m\right]\gamma^\mu\left[i((1-x_1)\not{k}_1 + x_2\not{k}_2)+m\right]\gamma_\nu \\ &\quad +\text{terms linear in } p. \end{aligned}$$

The terms linear in $p$ evaluate to zero in the integral, as they are odd, so we can discard them. The first term can be evaluated using the identity $\gamma^\nu\not{a}\not{b}\not{c}\gamma_\nu = -2\not{c}\not{b}\not{a}$, resulting in

$$-\gamma^\nu\not{p}\gamma^\mu\not{p}\gamma_\nu = -2\not{p}\not{p}\gamma^\mu + 4\not{p}p^\mu = -2p^2\gamma^\mu + 4p^\nu p^\mu\gamma_\nu. \tag{18}$$

This last term is again odd in the individual components of the momentum integral, and so reduces to $\frac{p^2g^{\nu\mu}}{4}\gamma_\nu$ inside the integral. So, the contribution to the integrand from the first term is

$$-p^2\gamma^\mu. \tag{19}$$

We see that this term contributes to $F_1$, and we know by the Ward identity that $F_1(0)$ is zero. We can disregard this term here. The second term is $p-$independent, convergent in the ultraviolet, and contains a contribution to $F_2$. We use the identities

$$\begin{aligned} \gamma^\nu\gamma^\alpha\gamma^\beta\gamma_\nu &= 4g^{\alpha\beta}, \\ \gamma^\nu\gamma^\alpha\gamma_\nu &= -2\gamma^\alpha, \end{aligned}$$

and so the relevant contribution to $N^\mu$ is

$$\begin{aligned} N^\mu &= -2\left[-ix_2\not{k}_2 + i(1-x_1)\not{k}_1\right]\gamma^\mu\left[-ix_1\not{k}_1 + i(1-x_2)\not{k}_2\right] \\ &\quad +4m\left[i(1-2x_1)k_1^\mu + i(1-2x_2)k_2^\mu\right] - 2m^2\gamma^\mu. \end{aligned}$$

We discard the last term, which again contributes to $F_1$. We again use the fact that $\Gamma^\mu$ appears in the combination $\bar{u}(k_2)\Gamma^\mu u(k_1)$ for on-shell spinors, and that $\bar{u}\not{k}_2 = -im\bar{u}$, $\not{k}_1 u = -imu$, for on-shell solutions. So, the relevant part of $N^\mu$ in the integrand can be written as

$$\begin{aligned} N^\mu &= -2\left[-x_2m + i(1-x_1)\not{k}_1\right]\gamma^\mu\left[-x_1m + i(1-x_2)\not{k}_2\right] \\ &\quad +4im\left[(1-2x_1)k_1^\mu + (1-2x_x)k_2^\mu\right]. \end{aligned}$$

Using the identity $\not{k}\gamma^\mu = -\gamma^\mu\not{k} + 2k^\mu$, and again retaining only the parts contributing to $F_2$, the relevant part of $N^\mu$ reduces finally to

$$N^\mu = 2im(x_1+x_2)(1-x_1-x_2)(k_1^\mu + k_2^\mu). \tag{20}$$

Thus, we find

$$\Gamma_1^\mu = \gamma^\mu(\dots) + (k_1^\mu + k_2^\mu)B, \tag{21}$$

with

$$B = 2e^3\int_0^1\int_0^1\int_0^1 dx_1dx_2dx_3\delta(x_1+x_2+x_3-1)\int\frac{d^4p}{(2\pi)^4}\frac{2im(x_1+x_2)(1-x_1-x_2)}{\left(p^2+x_1x_2q^2+(x_1+x_2)^2m^2-i\epsilon\right)^3}. \tag{22}$$

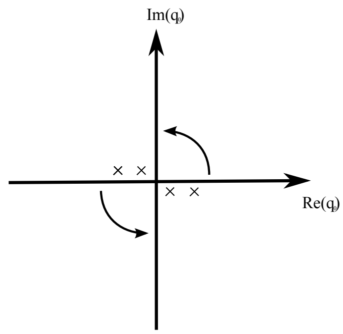



Figure 1: Illustration of the Wick rotation of the variable $q_0$.

Applying the Wick rotation, $p^0 \equiv ip_E^4$, $d^4p = id^4p_E$, we can explicitly evaluate

$$\int \frac{d^4p_E}{(2\pi)^4} \frac{1}{(p_E^2 + \Delta)} = \frac{1}{32\pi^2\Delta}, \tag{23}$$

and so

$$B = -4me^3 \frac{1}{32\pi^2} \int_0^1 \int_0^1 \int_0^1 dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 1) \frac{x_3(1-x_3)}{x_1x_2q^2 + (1-x_3)^2m^2}, \tag{24}$$

and, finally, for the form-factor $F_2$ we obtain the result

$$\begin{aligned} F_2(0) &= -\frac{2m}{e}B(q^2 = 0) = \frac{e^2}{4\pi^2}\int_0^1 dx_3 \int_0^{1-x_3} dx_2 \frac{x_3}{1-x_3} \\ &= \frac{e^2}{4\pi^2}\int_0^1 dx_3 x_3 = \frac{e^2}{8\pi^2} = \frac{\alpha}{2\pi}, \end{aligned}$$

where $\alpha \equiv \frac{e^2}{4\pi} \sim \frac{1}{137}$, and so

$$g = 2 + 2F_2(0) = 2 + \frac{\alpha}{\pi}. \tag{25}$$

$a_e = \frac{g-2}{2} = \frac{\alpha}{2\pi} = 0.0011614..$ at one-loop level. Experimentally, $a_e = 0.00115965218073(28)$ (Gabrielse 2008). Theoretically, the result is decomposed as

$$F_2(0) = \frac{\alpha}{2\pi} + a_2\left(\frac{\alpha}{\pi}\right)^2 + a_3\left(\frac{\alpha}{\pi}\right)^3 + a_4\left(\frac{\alpha}{\pi}\right)^4, \tag{26}$$

where the second coefficient, $a_2$, consists of seven diagrams, and was calculated in 1957. The third coefficient consists of 72 diagrams, and was calculated in 1996. The fourth coefficient, $a_4$, consists of 891 diagrams.