8.324 Relativistic Quantum Field Theory II

4: GENERAL RENORMALIZATION THEORY

Let us recall some of the major results and methods of previous lectures:

1. In perturbation theory, bare and physical quantities are related by ultraviolet-divergent expressions:

$$m_{phys} = m_B + \delta m, \tag{1}$$

where m_{phys} is finite, δm is ultraviolet-divergent, and so m_B is necessarily ultraviolet-divergent.

2. We express the Lagrangian in terms of physical quantites, and separate it into

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_I + \mathscr{L}_{ct}, \tag{2}$$

where $\mathscr{L}_0$ is the canonically normalized free Lagrangian for physical fields and masses, $\mathscr{L}_I$ contains the interaction, again in terms of physical parameters, and $\mathscr{L}_{ct}$ contains the counterterms with ultraviolet-divergent coefficients. From $\mathscr{L}_0$, we obtain the propagators of the physical fields. $\mathscr{L}_I$ and $\mathscr{L}_{ct}$ give interaction vertices.

3. At the one-loop level, the self-energy is given by the effective two-point vertices: the 1PI two-point vertex of the interaction and the counter-term two-point vertex. The counterterms absorb ultraviolet divergences, and the finite parts of the counterterms are determined by renormalization conditions, which ensure the quantities in $\mathscr{L}_0 + \mathscr{L}_I$ are physical. The conditions constrain the self-energy and the effective vertices, and give a finite, uniquely-determined value for the counterterms.

There are several questions we need to consider:

1. Can counterterms remove higher order divergences in the self-energies and vertex corrections?
2. Can they remove ultraviolet divergences in generic physical observables?
3. How does the procedure work for a general theory?
4. What is the physics behind the success (or failure) of renormalization of ultraviolet divergences?

4.1: DEGREES OF DIVERGENCES

Given a generic 1PI diagram M in quantum electrodynamics, or any theory, how can we tell whether it is ultraviolet divergent or not? We begin by introducing the superficial degree of divergence, D. This is defined by

$$\begin{aligned} D \quad = \quad & \text{number of factors of internal momentum in the numerator } - \\ & \text{number of factors of internal momentum in the denominator} \end{aligned}$$

If we let all the loop momenta go to infinity with a common factor $S \longrightarrow \infty$, then

$$M \sim \int^{\Lambda} dS\, S^{D-1} \sim \begin{cases} \Lambda^D & D > 0, \\ \log \Lambda & D = 0, \\ \text{finite} & D < 0. \end{cases} \tag{3}$$

It is the superficial degree because it gives a rough indication of the behaviour. We will mention caveats and how to deal with them later in the lecture. We will now find an explicit expression for D in the case of quantum electrodynamics. Introduce

$$\begin{aligned} E_e &\equiv \text{Number of external electron lines,} \\ E_\gamma &\equiv \text{Number of external photon lines,} \\ I_e &\equiv \text{Number of internal electron lines,} \\ I_\gamma &\equiv \text{Number of internal photon lines,} \\ V &\equiv \text{Number of vertices,} \\ L &\equiv \text{Number of loops.} \end{aligned}$$

Then we have

$$D = 4L - I_e - 2I_\gamma. \tag{4}$$

We can express I_e, I_γ and L in terms of E_e, E_γ and V:

$$\begin{aligned} 2I_\gamma + E_\gamma &= V, \\ 2I_e + E_e &= 2V, \\ I_e + I_\gamma - (V-1) &= L, \end{aligned}$$

and so $I_\gamma = \frac{1}{2}(V - E_\gamma)$, $I_e = V - \frac{E_e}{2}$ and we find

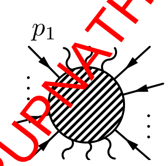
$$D = 4 - E_\gamma - \frac{3}{2}E_e. \tag{5}$$

We see that D only depends on the number of external legs, not on the internal structure of a diagram. In order to have ultraviolet divergences, we require $D \geq 0$, and so

$$E_\gamma + \frac{3}{2}E_e \leq 4. \tag{6}$$

Therefore, only a finite number of external lines can yield superficially divergent integrals. So we have a finite number of classes of divergent diagrams. They precisely correspond to the counter terms we discussed earlier. We will enumerate them in the next section. Let us now generalize (4) to a general theory. First, we provide an alternative derivation:

$M\delta(p_1 + \ldots) \quad =$



$= \quad$ Fourier transform of $\left\langle \psi(x_1)\ldots\psi(x_{E_e})A(y_1)\ldots A(y_{E_\gamma})\right\rangle$ with external legs amputated. (7)

Hence,

$$M \sim \Lambda^D e^V \tag{8}$$

where e is the coupling constant. We have that $[M] = D$, as $[e] = 0$, and therefore

$$\begin{aligned} [M] - 4 &= \left[\frac{\int d^4x_1 \ldots d^4y_1 \ldots \langle\psi(x_1)\ldots A(y_1)\ldots\rangle}{\int d^4x_1 \left\langle\bar\psi(x_1)\psi(x_1)\right\rangle \ldots \int d^4y_1 \left\langle A(y_1)A(y_1)\right\rangle \ldots}\right] \\ &= -E_e\,[\psi] - E_\gamma\,[A]\,. \end{aligned}$$

We note that $[\psi] = \frac{3}{2}$ and $[A] = 1$, and hence

$$D = [M] = 4 - \frac{3}{2}E_e - E_\gamma. \tag{9}$$

Now, for a general theory in d spacetime dimensions, the field content is given by ϕ_f, $f = 1, 2, \ldots$, where f labels the field type. $[\phi_f] = \Delta_f$ and $\Delta_f \geq 0$ in all physical theories. We have interaction vertices of type i, $i = 1, 2, \ldots$, contributing a term of the form

$$\lambda_i(\partial)^{n_i}\prod_f \phi_f^{n_{if}}. \tag{10}$$

Here, λ_i is the coupling constant, with dimension

$$[\lambda_i] = \delta_i = d - n_i - \sum_f n_{if}\Delta_f. \tag{11}$$

Now consider a 1PI diagram in such a theory:

$$E_f \equiv \text{number of external lines of } \phi_f,$$
$$V_i \equiv \text{number of vertices of type } i.$$

Then $M \sim \Lambda^D \prod_i \lambda_i^{V_i}$ and so $D = [M] - \sum_i V_i\delta_i$. Again,

$$[M] - d = \sum_f E_f\Delta_f \tag{12}$$

and so the general expression for the superficial degree of divergence is given by

$$D = d - \sum_f E_f\Delta_f - \sum_i V_i\delta_i. \tag{13}$$

Diagrams which are ultraviolet divergent satisfy

$$\sum_f E_f\Delta_f + \sum_i V_i\delta_i \leq d. \tag{14}$$

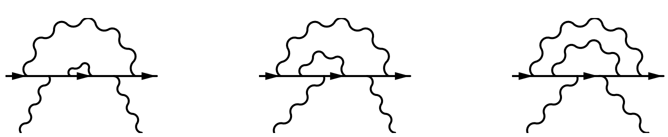
We note that $\Delta_f \geq 0$ in all physical theories. We can now divide all theories into

1. All $\delta_i > 0$: there are only a finite number of superficially divergent diagrams. These are **super-renormalizable** theories.
2. All $\delta_i \geq 0$: there are a finite number of classes of divergent diagrams. These are **renormalizable** theories.
3. There exists at least one $\delta_i < 0$. In these theories, increasing V_i means increaing D, so all amplitudes are divergent at high enough orders. These are **non-renormalizable** theories.

These terms also apply to individual interactions for a vertex of type i:

1. $\delta_i > 0$: super-renormalizable, **relevant** interaction.
2. $\delta_i = 0$: renormalizable, **marginal** interaction.
3. $\delta_i < 0$: non-renormalizable, **irrelevant** interaction.

Note that a diagram can be superficially convergent but divergent because of a divergent supdiagram. For example, in quantum electrodynamics, in the case $E_e = 2$, $E_\gamma = 2$, in which we have that $D = -1$



(15)

Figure 1: The first two diagrams with $E_e = 2$, $E_\gamma = 2$ both diverge because they contain divergent subdiagrams, whereas the third diagram is finite.