8.324 Relativistic Quantum Field Theory II

4.2: CANCELLATION OF DIVERGENCES

4.2.1: General Structure

Consider a generic divergent diagram M of degree D; that is,

$$M \sim \int^{\Lambda} ds\, s^{D-1} \tag{1}$$

if all loop momenta are taken proportional to s. Generically, internal propagators have the form

$$\frac{1}{(as+p)^{\alpha} + \dots} \sim \frac{1}{s^{\alpha}} \quad \alpha = \begin{cases} 2 & \text{scalar or vector,} \\ 1 & \text{fermion,} \end{cases} \tag{2}$$

for large s, where a is a numerical constant and p is a combination of the external momenta. Differentiating M, with respect to p, n times gives a term proportional to

$$\frac{1}{(as+p)^{\alpha+n}} \sim \frac{1}{s^{\alpha+n}}, \tag{3}$$

and so $D+1$ derivatives with respect to the external momenta will make M finite. This means that

$$M(p) = M_0 + M_1 p + \dots M_D p^D + \text{finite terms}, \tag{4}$$

where the argument p of the function represents the collection of external momenta, we have suppressed the index structure, and M_0,$M_1, \dots, M_D$ are potentially divergent constants. Suppose that M has E_f external lines of the field ϕ_f. Then divergences of $M(p)$ can be cancelled by counterterms of the form

$$\sum_{j=0}^{D} A_j (\partial)^j \prod_f \phi_f^{E_f} \tag{5}$$

where the A_j are divergent coefficients in order to cancel divergences in M_j. The index structure in $A_j \partial^j$ should match the suppressed index structure of $M(p)$.

4.2.2: Renormalization of Quantum Electrodynamics

In the case of quantum electrodynamics,

$$D = 4 - E_{\gamma} - \frac{3}{2} E_e, \tag{6}$$

and hence, ultraviolet-divergent diagrams are given by

$$E_{\gamma} + \frac{3}{2} E_e \leq 4. \tag{7}$$

We now enumerate these classes of diagrams:

1. $E_{\gamma} = E_e = 0$, $D = 4$: These are the vacuum diagrams, and give a divergent constant Λ_0. The divergence can be removed by including a constant Λ_{ct} counterterm in the Lagrangian, such that

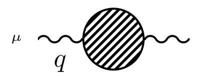
$$\Lambda_{phys} = \Lambda_0 - \Lambda_{ct}. \tag{8}$$

Λ_{phys} does not have any physical significance, unless we are dealing with a theory of gravity. Λ_{phys} is measured to be very small, and it acts as a cosmological constant.

2. $E_e = 1$, $E_\gamma = 0, 1, 2$ and $E_e = 0$, $E_\gamma = 1$: These terms are all zero because of Lorentz invariance.

3. $E_e = 0$, $E_\gamma = 2$, $D = 2$: This is the photon self-energy,

$$\Pi^{\mu\nu}(q) = \tag{9}$$



Expanding this term, we get

$$\Pi^{\mu\nu}(q) = c_1\eta^{\mu\nu} + c_2 q^2 \eta^{\mu\nu} + c_3 q^\mu q^\nu + \text{finite terms}. \tag{10}$$

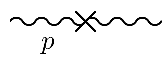
The Ward identity imposes

$$\Pi^{\mu\nu}(q) = q^2 P_T^{\mu\nu}(q)\Pi(q^2), \tag{11}$$

and so we have

$$\Pi(q^2) = C + \text{finite terms} \tag{12}$$

where C is the only divergent constant. The divergence in C can be absorbed in the counterterm vertex,

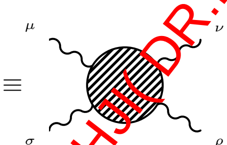


$$= (Z_3 - 1)F^{\mu\nu}F_{\mu\nu}. \tag{13}$$

4. $E_e = 0$, $E_\gamma = 3$, $D = 1$: This is zero by charge conjugation invariance, as will be shown on the problem set.

5. $E_e = 0$, $E_\gamma = 4$, $D = 0$: This is the four-point vertex,

$$M_{\mu\nu\rho\sigma} \equiv$$



$$= K(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) + \text{finite terms}, \tag{14}$$

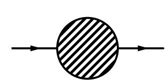
where K is a potentially divergent constant, and the form of the divergent term follows from symmetry. The Ward identity constrains that

$$q^\mu M_{\mu\nu\rho\sigma} = 0, \tag{15}$$

and so $K = 0$, and the four-point vertex is convergent.

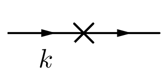
6. $E_e = 2$, $E_\gamma = 0$, $D = 1$: This is the fermion self-energy,

$$\Sigma(k) \equiv$$



$$= A + B\not{k} + \text{finite terms}. \tag{16}$$

Here, A and B are divergent constants, and they can be cancelled by the counterterm vertex,



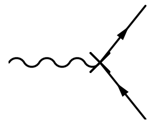
$$= (Z_2 - 1)(i\not{k} - m) + Z_2\delta m, \tag{17}$$

with the mass renormalization δm and the field strength renormalization Z_2 chosen appropriately.

7. $E_e = 2$, $E_\gamma = 1$, $D = 0$: This is the vertex function,

$$\Gamma^\mu(k, k') = A\gamma^\mu + \text{finite terms}. \tag{18}$$

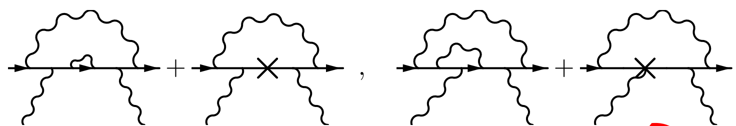
where A is a logarithmically divergent coefficient. This can be cancelled by the counterterm three-point vertex,



$$= -Z_1 e A_\mu \bar{\psi}\gamma^\mu\psi, \tag{19}$$

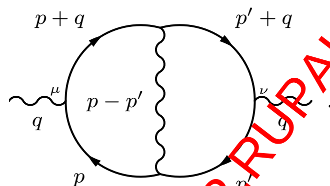
where the Ward identity imposes $Z_1 = Z_2$.

In summary, the counter terms we considered before are enough to cancel all of the superficial divergences to all orders in perturbation theory. We still need to consider diagrams with divergent sub-diagrams. Generically, these can be easily cancelled since divergent sub-graphs must also belong to those considered above. For example, the pairings



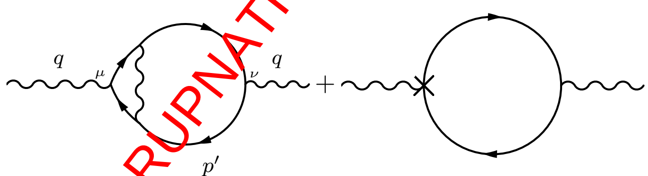
(20)

are both finite. We now need to consider the case of overlapping divergences. That is, when two divergent subgraphs share an internal line. These divergences are more difficult to deal with. For example, consider the diagram



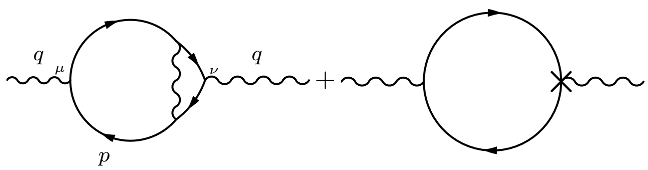
(21)

In the case that p is taken to be large, the pairing



(22)

gives a vertex correction at the vertex μ, leaving a residual self-energy divergence for the case $p' \longrightarrow \infty$. Similarly, if p' is taken to be large, the pairing

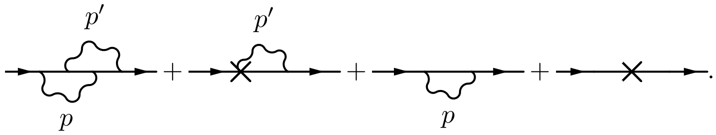


(23)

gives a vertex correction at the vertex ν, leaving a residual self-energy divergence for the case $p' \longrightarrow \infty$. The two procedures are not independent, and it is finally necessary to add a correction of the form

(24)

at the order $O(e^4)$ to cancel the divergences when both p and p' are large. Another example is given by the combination of diagrams



(25)

It can be proved to all orders, the content of the Bogoliubov-Parasiuk-Hepp-Zimermann theorem, that overlapping divergences can be canceled by counterterms corresponding to superficial divergences.