Chapter 11

Spinor Formulation of Relativistic Quantum Mechanics

11.1 The Lorentz Transformation of the Dirac Bispinor

We will provide in the following a new formulation of the Dirac equation in the chiral representation defined through (10.225–10.229). Starting point is the Lorentz transformation $\tilde{\mathcal{S}}(\vec{w}, \vec{\vartheta})$ for the bispinor wave function $\tilde{\Psi}$ in the chiral representation as given by (10.262). This transformation can be written

$$\tilde{\mathcal{S}}(\vec{z}) = \begin{pmatrix} a(\vec{z}) & 0 \\ 0 & b(\vec{z}) \end{pmatrix} \tag{11.1}$$

$$a(\vec{z}) = \exp\left(\frac{1}{2}\vec{z}\cdot\vec{\sigma}\right) \tag{11.2}$$

$$b(\vec{z}) = \exp\left(-\frac{1}{2}\vec{z}^{\,*}\cdot\vec{\sigma}\right) \tag{11.3}$$

$$\vec{z} = \vec{w} - i\,\vec{\vartheta}\,. \tag{11.4}$$

We have altered here slightly our notation of $\tilde{\mathcal{S}}(\vec{w}, \vec{\vartheta})$, expressing its dependence on $\vec{w}, \vec{\vartheta}$ through a complex variable $\vec{z}, \vec{z} \in \mathbb{C}^3$.

Because of its block-diagonal form each of the diagonal components of $\tilde{\mathcal{S}}(\vec{z})$, i.e., $a(\vec{z})$ and $b(\vec{z})$, must be two-dimensional irreducible representations of the Lorentz group. This fact is remarkable since it implies that the representations provided through $a(\vec{z})$ and $b(\vec{z})$ are of lower dimension then the four-dimensional natural representation[1] $L(\vec{w}, \vec{\vartheta})^{\mu}{}_{\nu}$. The lower dimensionality of $a(\vec{z})$ and $b(\vec{z})$ implies, in a sense, that the corresponding representation of the Lorentz group is more basic than the natural representation and may serve as a building block for all representations, in particular, may be exploited to express the Lorentz-invariant equations of relativistic quantum mechanics. This is, indeed, what will be achieved in the following.

We will proceed by building as much as possible on the results obtained sofar in the chiral representation of the Dirac equation. We will characterize the space on which the transformations $a(\vec{z})$

[1] We will see below that the representations $a(\vec{z})$ and $b(\vec{z})$ are, in fact, isomorphic to the natural representation, i.e., different $L(\vec{w}, \vec{\vartheta})^{\mu}{}_{\nu}$ correspond to different $a(\vec{z})$ and $b(\vec{z})$.

and $b(\vec{z})$ act, the so-called spinor space, will establish the map between $L(\vec{w},\vec{\vartheta})^{\mu}{}_{\nu}$ and $a(\vec{z})$, $b(\vec{z})$, express 4-vectors A^{μ}, A_{ν}, the operator ∂_{μ} and the Pauli matrices $\vec{\sigma}$ in the new representation and, finally, formulate the Dirac equation, neutrino equation, and the Klein–Gordon equation in the spinor representation.

A First Characterization of the Bispinor States

We note that in case $\vec{w} = 0$ the Dirac transformations are pure rotations. In this case $a(\vec{z})$ and $b(\vec{z})$ are identical and read

$$a(i\,\vec{\vartheta}) \;=\; b(i\,\vec{\vartheta}) \;=\; \exp\left(-\frac{1}{2}\vec{\vartheta}\cdot\vec{\sigma}\right)\,, \qquad \theta \;\in\; \mathbb{R}^3\,. \tag{11.5}$$

The transformations in this case, i.e., for $\vec{z} = i\vec{\vartheta}, \vec{\vartheta} \in \mathbb{R}^3$, are elements of SU(2) and correspond, in fact, to the rotational transformations of spin-$\frac{1}{2}$ states as described by $D^{(\frac{1}{2})}_{m\,m'}(\vec{\vartheta})$, usually expressed as product of rotations and of functions of Euler angles α, β, γ (see Chapter 5). For $\vec{\vartheta} = (0,\beta,0)^T$ the transformations are

$$a(i\,\beta\,\hat{e}_2) \;=\; b(i\,\beta\,\hat{e}_2) \;=\; \left(\,d^{(\frac{1}{2})}_{mm'}(\beta)\,\right) \;=\; \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}. \tag{11.6}$$

as given by (5.243). This characterization allows one to draw conclusions regarding the state space in which $a(\vec{z})$ and $b(\vec{z})$ operate, namely, a space of vectors $\binom{\text{state1}}{\text{state2}}$ for which holds

$$\begin{pmatrix} \text{state 1} & \sim & |\frac{1}{2},+\frac{1}{2}\rangle \\ \text{state 2} & \sim & |\frac{1}{2},-\frac{1}{2}\rangle \end{pmatrix} \quad \vec{z} \;=\; i\vec{\vartheta},\ \vec{\vartheta} \in \mathbb{R}^3 \tag{11.7}$$

where "$\sim$" stands for "*transforms like*". Here $|\frac{1}{2},\pm\frac{1}{2}\rangle$ represents the familiar spin–$\frac{1}{2}$ states. Since $a(\vec{z})$ acts on the first two components of the solution $\tilde{\Psi}$ of the Dirac equation, and since $b(\vec{z})$ acts on the third and fourth component of $\tilde{\Psi}$ one can characterize $\tilde{\Psi}$

$$\begin{pmatrix} \tilde{\Psi}_1(x^{\mu}) & \sim & |\frac{1}{2},+\frac{1}{2}\rangle \\ \tilde{\Psi}_2(x^{\mu}) & \sim & |\frac{1}{2},-\frac{1}{2}\rangle \\ \tilde{\Psi}_3(x^{\mu}) & \sim & |\frac{1}{2},+\frac{1}{2}\rangle \\ \tilde{\Psi}_4(x^{\mu}) & \sim & |\frac{1}{2},-\frac{1}{2}\rangle \end{pmatrix} \quad \vec{z} \;=\; i\vec{\vartheta},\ \vec{\vartheta} \in \mathbb{R}^3\,. \tag{11.8}$$

We like to stress that there exists, however, a distinct difference in the transformation behaviour of $\tilde{\Psi}_1(x^{\mu}), \tilde{\Psi}_2(x^{\mu})$ and $\tilde{\Psi}_3(x^{\mu}), \tilde{\Psi}_4(x^{\mu})$ in case $\vec{z} = \vec{w} + i\vec{\vartheta}$ for $\vec{w} \neq 0$. In this case holds $a(\vec{z}) \neq b(\vec{z})$ and $\tilde{\Psi}_1(x^{\mu}), \tilde{\Psi}_2(x^{\mu})$ transform according to $a(\vec{z})$ whereas $\tilde{\Psi}_3(x^{\mu}), \tilde{\Psi}_4(x^{\mu})$ transform according to $b(\vec{z})$.

Relationship Between $a(\vec{z})$ and $b(\vec{z})$

The transformation $b(\vec{z})$ can be related to the conjugate complex of the transformation $a(\vec{z})$, i.e., to

$$a^*(\vec{z}) \;=\; \exp\left(\frac{1}{2}\vec{z}^{\,*}\cdot\vec{\sigma}^{\,*}\right) \tag{11.9}$$

where $\vec{\sigma}^{\,*} = (\sigma_1^*, \sigma_2^*, \sigma_3^*)$. One can readily verify [c.f. (5.224)]

$$\sigma_1 \;=\; \sigma_1^* \,, \qquad \sigma_2 \;=\; -\,\sigma_2^* \,, \qquad \sigma_3 \;=\; \sigma_3^* \,. \tag{11.10}$$

From this one can derive

$$b(\vec{z}) \;=\; \epsilon\, a^*(\vec{z})\, \epsilon^{-1} \tag{11.11}$$

where

$$\epsilon \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \qquad \epsilon^{-1} \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{11.12}$$

To prove (11.11) one first demonstrates that for ϵ and ϵ^{-1} as given in (11.12) does, in fact, hold $\epsilon\,\epsilon^{-1} = 1\!\!1$. One notices then, using $\epsilon f(a)\epsilon^{-1} = f(\epsilon a \epsilon^{-1})$,

$$\epsilon\, a^*(\vec{z})\epsilon^{-1} \;=\; \exp\left[\frac{1}{2}\,\vec{z}^{\,*}\left(\epsilon\vec{\sigma}^{\,*}\epsilon^{-1}\right)\right] . \tag{11.13}$$

Explicit matrix multiplication using (5.224, 11.10, 11.12) yields

$$\begin{aligned} \epsilon\sigma_1\epsilon^{-1} \;&=\; -\sigma_1 \;&=\; -\sigma_1^* \\ \epsilon\sigma_2\epsilon^{-1} \;&=\; \sigma_2 \;&=\; -\sigma_2^* \\ \epsilon\sigma_3\epsilon^{-1} \;&=\; -\sigma_3 \;&=\; -\sigma_3^* \,, \end{aligned} \tag{11.14}$$

or in short

$$\epsilon\vec{\sigma}\epsilon^{-1} \;=\; -\vec{\sigma}^{\,*} \,. \tag{11.15}$$

Similarly, one can show

$$\epsilon^{-1}\vec{\sigma}\epsilon \;=\; -\vec{\sigma}^{\,*} \,, \tag{11.16}$$

a result to be used further below. Hence, one can express

$$\epsilon\, a^*(\vec{z})\epsilon^{-1} \;=\; \exp\left(-\frac{1}{2}\,\vec{z}^{\,*}\cdot\vec{\sigma}\right) \;=\; b(\vec{z}) \,. \tag{11.17}$$

We conclude, therefore, that the transformation (11.1) can be written in the form

$$\tilde{\mathcal{S}}(\vec{z}) \;=\; \begin{pmatrix} a(\vec{z}) & 0 \\ 0 & \epsilon\, a^*(\vec{z})\, \epsilon^{-1} \end{pmatrix} \tag{11.18}$$

with $a(\vec{z})$ given by (11.2, 11.4) and ϵ, ϵ^{-1} given by (11.12). This demonstrates that $a(\vec{z})$ is the transformation which characterizes both components of $\tilde{\mathcal{S}}(\vec{z})$.

Spatial Inversion

One may question from the form of $\tilde{\mathcal{S}}(\vec{z})$ why the Dirac equation needs to be four-dimensional, featuring the components $\tilde{\Psi}_1(x^\mu), \tilde{\Psi}_2(x^\mu)$ as well as $\tilde{\Psi}_3(x^\mu)$, $\tilde{\Psi}_4(x^\mu)$ even though these pairs of components transform independently of each other. The answer lies in the necessity that application of spatial inversion should transform a solution of the Dirac equation into another possible solution of the Dirac equation. The effect of inversion on Lorentz transformations is, however, that they alter $\vec{w}$ into $-\vec{w}$, but leave rotation angles $\vec{\vartheta}$ unaltered.

Let $\mathcal{P}$ denote the representation of spatial inversion in the space of the wave functions $\tilde{\Psi}$. Obviously, $\mathcal{P}^2 = \mathbb{1}$, i.e., $\mathcal{P}^{-1} = \mathcal{P}$. The transformation $\tilde{\mathcal{S}}(\vec{z})$ in the transformed space is then

$$\mathcal{P}\,\tilde{\mathcal{S}}(\vec{w} + i\vec{\vartheta})\,\mathcal{P} \;=\; \tilde{\mathcal{S}}(-\vec{w} + i\vec{\vartheta}) \;=\; \begin{pmatrix} b(\vec{z}) & 0 \\ 0 & a(\vec{z}) \end{pmatrix} , \tag{11.19}$$

i.e., the transformations $a(\vec{z})$ and $b(\vec{z})$ become interchanged. This implies

$$\mathcal{P} \begin{pmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \\ \tilde{\Psi}_3 \\ \tilde{\Psi}_4 \end{pmatrix} \;=\; \begin{pmatrix} \tilde{\Psi}_3 \\ \tilde{\Psi}_4 \\ \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{pmatrix} . \tag{11.20}$$

Obviously, the space spanned by only two of the components of $\tilde{\Psi}$ is not invariant under spatial inversion and, hence, does not suffice for particles like the electron which obey inversion symmetry. However, for particles like the neutrinos which do not obey inversion symmetry two components of the wave function are sufficient. In fact, the Lorentz invariant equation for neutrinos is only 2–dimensional.

11.2 Relationship Between the Lie Groups SL(2,C) and SO(3,1)

We have pointed out that $a(i\vec{\vartheta}), \vec{\vartheta} \in \mathbb{R}^3$, which describes pure rotations, is an element of SU(2). However, $a(\vec{w} + i\vec{\vartheta})$ for $\vec{w} \neq 0$ is an element of

$$\mathrm{SL}(2,\mathbb{C}) \;=\; \{\, M,\ M \text{ is a complex } 2\times 2\text{–matrix}, \det(M) \;=\; 1 \,\} . \tag{11.21}$$

One can verify this by evaluating the determinant of $a(\vec{z})$

$$\det(\,a(\vec{z})\,) \;=\; \det\left(e^{\frac{1}{2}\vec{z}\cdot\vec{\sigma}} \right) \;=\; e^{\mathrm{tr}(\frac{1}{2}\vec{z}\cdot\vec{\sigma})} \;=\; 1 \tag{11.22}$$

which follows from the fact that for any complex, non-singular matrix M holds[2]

$$\det\left(e^{M} \right) \;=\; e^{\mathrm{tr}(M)} \tag{11.23}$$

and from [c.f. (5.224)]

$$\mathrm{tr}(\,\sigma_j\,) \;=\; 0\,, \qquad j \;=\; 1,2,3\,. \tag{11.24}$$

Exercise 11.2.1: Show that $SL(2,\mathbb{C})$ defined in (11.21) together with matrix multiplication as the binary operation forms a group.

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[2]The proof of this important property is straightforward in case of hermitian M (see Chapter 5). For the general case the proof, based on the Jordan–Chevalley theorem, can be found in G.G.A. Bäuerle and E.A. de Kerf *Lie Algebras, Part* (Elsevier, Amsterdam, 1990), Exercise 1.10.3.

Mapping A^μ onto matrices $M(A^\mu)$

We want to establish now the relationship between $SL(2,\mathbb{C})$ and the group $\mathcal{L}_+^\uparrow$ of proper, orthochronous Lorentz transformations. Starting point is a bijective map between $\mathbb{R}^4$ and the set of two-dimensional hermitian matrices defined through

$$M(A^\mu) \;=\; \sigma_\mu \, A^\mu \tag{11.25}$$

where

$$\sigma_\mu \;=\; \left(\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\sigma_0}, \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_1}, \underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_2}, \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_3} \right) . \tag{11.26}$$

The quantity σ_μ thus defined does not transform like a covariant 4–vector. In fact, one wishes that the definition (11.25) of the matrix $M(A^\mu)$ is independent of the frame of reference, i.e., in a transformed frame should hold

$$\sigma_\mu A^\mu \quad \xrightarrow{L^\rho{}_\nu} \quad \sigma_\mu A'^\mu \tag{11.27}$$

Straightforward transformation into another frame of reference would replace σ_μ by σ'_μ. Using $A^\mu = (L^{-1})^\mu{}_\nu A'^\nu$ one would expect in a transformed frame to hold

$$\sigma_\mu A^\mu \xrightarrow{L^\rho{}_\nu} \sigma'_\mu \, (L^{-1})^\mu{}_\nu \, A'^\nu \, . \tag{11.28}$$

Consistency of (11.28) and (11.27) requires then

$$L_\nu{}^\mu \sigma'_\mu \;=\; \sigma_\nu \tag{11.29}$$

where we used (10.76). Noting that for covariant vectors according to (10.75) holds $a'_\nu = L_\nu{}^\mu a_\mu$ one realizes that σ_μ transforms *inversely* to covariant 4–vectors. We will prove below [cf. (11.135)] this transformation behaviour.
$M(A^\mu)$ according to (11.25) can also be written

$$M(A^\mu) \;=\; \begin{pmatrix} A^0 + A^3 & A^1 - iA^2 \\ A^1 + iA^2 & A^0 - A^3 \end{pmatrix} . \tag{11.30}$$

Since the components of A^μ are real, the matrix $M(A^\mu)$ is hermitian as can be seen from inspection of (11.26) or from the fact that the matrices $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ are hermitian. The function $M(A^\mu)$ is bijective, in fact, one can provide a simple expression for the inverse of $M(A^\mu)$

$$M' \;=\; M(A^\mu) \qquad \leftrightarrow \qquad A^\mu \;=\; \frac{1}{2}\,\mathrm{tr}\left(M' \, \sigma_\mu \right) \, . \tag{11.31}$$

Exercise 11.2.2: Show that $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ provide a linear–independent basis for the space of hermitian 2×2–matrices. Argue why $M(A^\mu) = \sigma_\mu A^\mu$ provides a bijective map. Demonstrate that (11.31) holds.

The following important property holds for $M(A^\mu)$

$$\det\left(M(A^\mu) \right) \;=\; A^\mu A_\mu \tag{11.32}$$

which follows directly from (11.30).

Transforming the matrices $M(A^\mu)$

We define now a transformation of the matrix $M(A^\mu)$ in the space of hermitian 2×2–matrices

$$M \xrightarrow{a} M' = a\, M\, a^\dagger \,, \qquad a \in SL(2, \mathbb{C}) \,. \tag{11.33}$$

This transformation conserves the hermitian property of M since

$$(M')^\dagger = (\, a\, M\, a^\dagger\,)^\dagger = (a^\dagger)^\dagger M^\dagger a^\dagger = a\, M\, a^\dagger = M' \tag{11.34}$$

where we used the properties $M^\dagger = M$ and $(a^\dagger)^\dagger = a$. Due to $\det(a) = 1$ the transformation (11.33) conserves the determinant of M. In fact, it holds for the matrix M' defined through (11.33)

$$\begin{aligned} \det(M') &= \det(\, a\, M\, a^\dagger\,) = \det(a)\det(a^\dagger)\det(M) \\ &= [\det(a)]^2 \det(M) = \det(M) \,. \end{aligned} \tag{11.35}$$

We now apply the transformation (11.33) to $M(A^\mu)$ describing the action of the transformation in terms of transformations of A^μ. In fact, for any $a \in$ SL$(2, \mathbb{C})$ and for any A^μ there exists an A'^μ such that

$$M(A'^\mu) = a\, M(A^\mu)\, a^\dagger \,. \tag{11.36}$$

The suitable A'^μ can readily be constructed using (11.31). Accordingly, any $a \in$ SL$(2, \mathbb{C})$ defines the transformation [c.f. (11.31)]

$$A^\mu \xrightarrow{a} A'^\mu = \frac{1}{2}\,\mathrm{tr}\left(\, a\, M(A^\mu)\, a^\dagger \sigma_\mu \right) \,. \tag{11.37}$$

Because of (11.32, 11.35) holds for this transformation

$$A'^\mu A'_\mu = A^\mu A_\mu \tag{11.38}$$

which implies that (11.37) defines actually a Lorentz transformation. The linear character of the transformation becomes apparent expressing A'^μ as given in (11.37) using (11.25)

$$A'^\mu = \frac{1}{2}\,\mathrm{tr}\left(\, a\, \sigma_\nu a^\dagger \sigma_\mu \right)\, A^\nu \tag{11.39}$$

which allows us to express finally

$$A'^\mu = L(a)^\mu{}_\nu\, A^\nu \,; \qquad L(a)^\mu{}_\nu = \frac{1}{2}\,\mathrm{tr}\left(\, a\, \sigma_\nu a^\dagger \sigma_\mu \right) \,. \tag{11.40}$$

Exercise 11.2.3: Show that $L(a)^\mu{}_\nu$ defined in (11.40) is an element of $\mathcal{L}_+^\uparrow$.

$L(a)^\mu{}_\nu$ provides a homomorphism

We want to demonstrate now that the map between SL(2, ℂ) and SO(3,1) defined through $L(a)^\mu{}_\nu$ [cf. (11.40)] respects the group property of SL(2, ℂ) and of SO(3,1), i.e.,

$$\bar{L}^\mu{}_\rho \;=\; \underbrace{L(a_1)^\mu{}_\nu\, L(a_2)^\mu{}_\rho}_{\substack{\text{product}\\ \text{in SO(3,1)}}} \;=\; L(\underbrace{a_1\, a_2}_{\substack{\text{product}\\ \text{in SL(2,}\mathbb{C}\text{)}}})^\mu{}_\rho \tag{11.41}$$

For this purpose one writes using $\operatorname{tr}(AB) = \operatorname{tr}(BA)$

$$\begin{aligned} L(a_1)^\mu{}_\nu\, L(a_2)^\mu{}_\rho &= \sum_\nu \frac{1}{2}\operatorname{tr}\left(a_1\sigma_\nu a_1^\dagger\sigma_\mu\right)\frac{1}{2}\operatorname{tr}\left(a_2\sigma_\rho a_2^\dagger\sigma_\nu\right) \\ &= \sum_\nu \frac{1}{2}\operatorname{tr}\left(\sigma_\nu a_1^\dagger\sigma_\mu a_1\right)\frac{1}{2}\operatorname{tr}\left(a_2\sigma_\rho a_2^\dagger\sigma_\nu\right) . \end{aligned} \tag{11.42}$$

Defining

$$\Gamma \;=\; a_1^\dagger\sigma_\mu a_1 \;, \qquad \Gamma' \;=\; a_2\sigma_\rho a_2^\dagger \tag{11.43}$$

and using the definition of $\bar{L}^\mu{}_\rho$ in (11.41) results in

$$\bar{L}^\mu{}_\rho \;=\; \frac{1}{4}\sum_{\substack{\nu,\alpha,\beta\\ \gamma,\delta}}(\sigma_\nu)_{\alpha\beta}\Gamma_{\beta\alpha}\,\Gamma'_{\gamma\delta}(\sigma_\nu)_{\delta\gamma} \;=\; \frac{1}{4}\sum_{\substack{\alpha,\beta\\ \gamma,\delta}} A_{\alpha\beta\gamma\delta}\Gamma_{\beta\alpha}\,\Gamma_{\gamma\delta} \tag{11.44}$$

where

$$A_{\alpha\beta\gamma\delta} \;=\; \sum_\nu (\sigma_\nu)_{\alpha\beta}(\sigma_\nu)_{\delta\gamma} \;. \tag{11.45}$$

One can demonstrate through direct evaluation

$$A_{\alpha\beta\gamma\delta} \;=\; \begin{cases} 2 & \alpha = \beta = \gamma = \delta = 1 \\ 2 & \alpha = \beta = \gamma = \delta = 2 \\ 2 & \alpha = \gamma = 1\,,\; \beta = \delta = 2 \\ 2 & \alpha = \gamma = 2\,,\; \beta = \delta = 1 \\ 0 & \text{else} \end{cases} \tag{11.46}$$

which yields

$$\begin{aligned} \bar{L}^\mu{}_\rho &= \frac{1}{2}\left(\Gamma_{11}\Gamma'_{11} + \Gamma_{22}\Gamma'_{22} + \Gamma_{12}\Gamma'_{21} + \Gamma_{21}\Gamma'_{12}\right) \;=\; \frac{1}{2}\operatorname{tr}\left(\Gamma\Gamma'\right) \\ &= \frac{1}{2}\operatorname{tr}\left(a_1^\dagger\sigma_\mu a_1\, a_2\sigma_\rho a_2^\dagger\right) \;=\; \frac{1}{2}\operatorname{tr}\left(\sigma_\mu a_1 a_2\sigma_\rho a_2^\dagger a_1^\dagger\right) \\ &= \frac{1}{2}\operatorname{tr}\left(a_1 a_2\sigma_\rho (a_1 a_2)^\dagger\sigma_\mu\right) \;=\; L(a_1 a_2)^\mu{}_\rho \;. \end{aligned} \tag{11.47}$$

This completes the proof of the homomorphic property of $L(a)^\mu{}_\nu$.

Generators for SL(2, $\mathbb{C}$) which correspond to $\vec{K}$, $\vec{J}$

The transformations $a \in \mathrm{SL}(2,\mathbb{C})$ as complex 2×2–matrices are defined through four complex or, correspondingly, eight real numbers. Because of the condition $\det(a) = 1$ only six independent real numbers actually suffice for the definition of a. One expects then that six generators G_j and six real coordinates f_j can be defined which allow one to represent a in the form

$$a \;=\; \exp\left(\sum_{j=1}^{6} f_j G_j\right) \,. \tag{11.48}$$

We want to determine now the generators of the transformation $a(\vec{z})$ as defined in (11.2) which correspond to the generators $K_1, K_2, K_3, J_1, J_2, J_3$ of the Lorentz transformations $L^{\mu}{}_{\nu}$ in the natural representations, i.e., correspond to the generators given by (10.47, 10.48). To this end we consider infinitesimal transformations and keep only terms of zero order and first order in the small variables. To obtain the generator of $a(\vec{z})$ corresponding to the generator K_1, denoted below as κ_1, we write (11.36)

$$M(L^{\mu}{}_{\nu}A^{\nu}) \;=\; a\,M(A^{\mu})\,a^{\dagger} \tag{11.49}$$

assuming (note that $g^{\mu}{}_{\nu}$ is just the familiar Kronecker $\delta_{\mu\nu}$)

$$\begin{aligned} L^{\mu}{}_{\nu} &= g^{\mu}{}_{\nu} \,+\, \epsilon\,(K_1)^{\mu}{}_{\nu} & (11.50)\\ a &= \mathbb{1} \,+\, \epsilon\,\kappa_1 \,. & (11.51)\end{aligned}$$

Insertion of $(K_1)^{\mu}{}_{\nu}$ as given in (10.48) yields for the l.h.s. of (11.49), noting the linearity of $M(A^{\mu})$,

$$\begin{aligned} M(A^{\mu} \,+\, \epsilon\,(K_1)^{\mu}{}_{\nu}\,A^{\nu}) &= M(A^{\mu}) \,+\, \epsilon\,M(\,(-A^1,-A^0,0,0)\,) \\ &= M(A^{\mu}) \,-\, \epsilon\,\sigma_0 A^1 \,-\, \epsilon\,\sigma_1 A^0 \end{aligned} \tag{11.52}$$

where we employed (11.25) in the last step. For the r.h.s. of (11.49) we obtain using (11.51)

$$\begin{aligned} &(\,\mathbb{1} \,+\, \epsilon\,\kappa_1\,)\,M(A^{\mu})\,(\,\mathbb{1} \,+\, \epsilon\,\kappa_1^{\dagger}\,) \\ &\quad= M(A^{\mu}) \,+\, \epsilon\,(\,\kappa_1 M(A^{\mu}) \,+\, M(A^{\mu})\kappa_1^{\dagger}\,) \,+\, O(\epsilon^2) \\ &\quad= M(A^{\mu}) \,+\, \epsilon\,(\,\kappa_1\sigma_{\mu} \,+\, \sigma_{\mu}\kappa_1^{\dagger}\,)\,A^{\mu} \,+\, O(\epsilon^2) \,. \end{aligned} \tag{11.53}$$

Equating (11.52) and (11.53) results in the condition

$$\sigma_0 A^1 \,-\, \sigma_1 A^0 \;=\; (\,\kappa_1\sigma_{\mu} \,+\, \sigma_{\mu}\,\kappa_1^{\dagger}\,)\,A^{\mu} \,. \tag{11.54}$$

This reads for the four cases $A^{\mu} = (1,0,0,0)$, $A^{\mu} = (0,1,0,0)$, $A^{\mu} = (0,0,1,0)$, $A^{\mu} = (0,0,0,1)$

$$\begin{aligned} -\,\sigma_1 &= \kappa_1\sigma_0 \,+\, \sigma_0\kappa_1^{\dagger} \;=\; \kappa_1 \,+\, \kappa_1^{\dagger} & (11.55)\\ \sigma_0 &= \kappa_1\sigma_1 \,+\, \sigma_1\kappa_1^{\dagger} & (11.56)\\ 0 &= \kappa_1\sigma_2 \,+\, \sigma_2\kappa_1^{\dagger} & (11.57)\\ 0 &= \kappa_1\sigma_3 \,+\, \sigma_3\kappa_1^{\dagger} \,. & (11.58)\end{aligned}$$

One can verify readily that

$$\kappa_1 \;=\; -\frac{1}{2}\,\sigma_1 \tag{11.59}$$

obeys these conditions. Similarly, one can show that the generators κ_2, κ_3 of $a(\vec{z})$ corresponding to K_2, K_3 and $\lambda_1, \lambda_2, \lambda_3$ corresponding to J_1, J_2, J_3 are given by

$$\kappa \;=\; -\frac{1}{2}\,\vec{\sigma}\,, \qquad \vec{\lambda} \;=\; \frac{i}{2}\,\vec{\sigma}\,. \tag{11.60}$$

We can, hence, state that the following two transformations are equivalent

$$\underbrace{L(\vec{w},\vec{\vartheta}) \;=\; e^{\vec{w}\cdot\vec{K} \,+\, \vec{\vartheta}\cdot\vec{J}}}_{\substack{\in\ \mathrm{SO}(3,1) \\ \text{acts on 4–vectors } A^{\mu}}} \quad , \quad \underbrace{a(\vec{w} - i\vec{\vartheta}) \;=\; e^{-\frac{1}{2}(\vec{w}\,-\,i\vec{\vartheta})\cdot\vec{\sigma}}}_{\substack{\in\ \mathrm{SL}(2,\mathbb{C}) \\ \text{acts on } spinors\ \phi^{\alpha} \in \mathbb{C}^2 \\ \text{(characterized below)}}} \tag{11.61}$$

This identifies the transformations $a(\vec{z} = \vec{w} - i\vec{\vartheta})$ as representations of Lorentz transformations, $\vec{w}$ describing boosts and $\vec{\vartheta}$ describing rotations.

Exercise 11.2.4: Show that the generators (11.60) of $a \in \mathrm{SL}(2,\mathbb{C})$ correspond to the generators $\vec{K}$ and $\vec{J}$ of $L^{\mu}{}_{\nu}$.

11.3 Spinors

Definition of contravariant spinors

We will now further characterize the states on which the transformation $a(\vec{z})$ and its conjugate complex $a^{*}(\vec{z})$ act, the so-called *contravariant spinors*. We consider first the transformation $a(\vec{z})$ which acts on a 2-dimensional space of states denoted by

$$\phi^{\alpha} \;=\; \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} \;\in\; \mathbb{C}^2\,. \tag{11.62}$$

According to our earlier discussion holds

ϕ^1 transforms under rotations ($\vec{z} = i\vec{\vartheta}$) like a spin–$\frac{1}{2}$ state $|\frac{1}{2}, +\frac{1}{2}\rangle$
ϕ^2 transforms under rotations ($\vec{z} = i\vec{\vartheta}$) like a spin–$\frac{1}{2}$ state $|\frac{1}{2}, -\frac{1}{2}\rangle$.

We denote the general ($\vec{z} = \vec{w} + i\vec{\vartheta}$) transformation by

$$\phi'^{\alpha} \;=\; a^{\alpha}{}_{\beta}\,\phi^{\beta} \;\stackrel{\text{def}}{=}\; a^{\alpha}{}_1\phi^1 \;+\; a^{\alpha}{}_2\phi^2\,, \quad \alpha \;=\; 1,2 \tag{11.63}$$

where we extended the summation convention of 4-vectors to spinors. Here

$$a(\vec{z}) \;=\; (\,a^{\alpha}{}_{\beta}\,) \;=\; \begin{pmatrix} a^1{}_1 & a^1{}_2 \\ a^2{}_1 & a^2{}_2 \end{pmatrix} \tag{11.64}$$

describes the matrix $a(\vec{z})$.

Definition of a scalar product

The question arises if for the states ϕ^α there exists a scalar product which is invariant under Lorentz transformations, i.e., invariant under transformations $a(\vec{z})$. Such a scalar product does, indeed, exist and it plays a role for spinors which is as central as the role of the scalar product $A^\mu A_\mu$ is for 4–vectors. To arrive at a suitable scalar product we consider first only rotational transformations $a(i\vec{\vartheta})$. In this case spinors ϕ^α transform like spin–$\frac{1}{2}$ states and an invariant, which can be constructed from products $\phi^1\chi^2$, etc., is the singlet state. In the notation developed in Chapter 5 holds for the singlet state

$$|\frac{1}{2},\frac{1}{2};0,0\rangle \;=\; \sum_{m=\pm\frac{1}{2}} (0,0|\frac{1}{2},m;\frac{1}{2},-m)\,|\frac{1}{2},m\rangle_1|\frac{1}{2},-m\rangle_2 \tag{11.65}$$

where $|\cdots\rangle_1$ describes the spin state of "particle 1" and $|\cdots\rangle_2$ describes the spin state of "particle 2" and $(0,0|\frac{1}{2},\pm\frac{1}{2};\frac{1}{2},\mp\frac{1}{2})$ stands for the Clebsch–Gordon coefficient. Using $(0,0|\frac{1}{2},\pm\frac{1}{2};\frac{1}{2},\mp\frac{1}{2}) = \pm 1/\sqrt{2}$ and equating the spin states of "particle 1" with the spinor ϕ^α, those of "particle 2" with the spinor χ^β one can state that the quantity

$$\Sigma \;=\; \frac{1}{\sqrt{2}}\left(\phi^1\chi^2 - \phi^2\chi^1\right) \tag{11.66}$$

should constitute a singlet spin state, i.e., should remain invariant under transformations $a(i\vec{\vartheta})$. In fact, as we will demonstrate below such states are invariant under general Lorentz transformations $a(\vec{z})$.

Definition of covariant spinors

Expression (11.66) is a bilinear form, invariant and as such has the necessary properties of a scalar[3] product. However, this scalar product is anti-symmetric, i.e., exchange of ϕ^α and χ^β alters the sign of the expression. The existence of a scalar product gives rise to the definition of a dual representation of the states ϕ^α denoted by ϕ_α. The corresponding states are defined through

$$\phi^1\chi^2 \;-\; \phi^2\chi^1 \;=\; \phi^1\chi_1 \;+\; \phi^2\chi_2 \tag{11.67}$$

It obviously holds

$$\chi_\alpha \;=\; \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \;=\; \begin{pmatrix} \chi^2 \\ -\chi^1 \end{pmatrix}\,. \tag{11.68}$$

We will refer to $\phi^\alpha, \chi^\beta, \ldots$ as *contravariant spinors* and to $\phi_\alpha, \chi_\beta, \ldots$ as *covariant spinors*. The relationship between the two can be expressed

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \;=\; \epsilon\begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} \tag{11.69}$$

$$\epsilon \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{11.70}$$

[3]'Scalar' implies invariance under rotations and is conventionally generalized to invariance under other symmetry trasnformations.

as can be verified from (11.68). The inverse of (11.69, 11.70) is

$$\begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} = \epsilon^{-1} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{11.71}$$

$$\epsilon^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{11.72}$$

Exercise 11.3.1: Show that for any non-singular complex 2×2–matrix M holds

$$\epsilon \, M \, \epsilon^{-1} = \det(M) \left(M^{-1} \right)^T$$

The matrices ϵ, ϵ^{-1} connecting contravariant and covariant spinors play the role of the metric tensors $g_{\mu\nu}, g^{\mu\nu}$ of the Minkowski space [cf. (10.10, 10.74)]. Accordingly, we will express (11.69, 11.70) and (11.71, 11.72) in a notation analogous to that chosen for contravariant and covariant 4–vectors [cf. (10.72)]

$$\phi_\alpha = \epsilon_{\alpha\beta} \, \phi^\beta \tag{11.73}$$

$$\phi^\alpha = \epsilon^{\alpha\beta} \, \phi_\beta \tag{11.74}$$

$$\epsilon_{\alpha\beta} = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{11.75}$$

$$\epsilon^{\alpha\beta} = \begin{pmatrix} \epsilon^{11} & \epsilon^{12} \\ \epsilon^{21} & \epsilon^{22} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{11.76}$$

The scalar product (11.67) will be expressed as

$$\phi^\alpha \chi_\alpha = \phi^1 \chi_1 + \phi^2 \chi_2 = \phi^1 \chi^2 - \phi^2 \chi^1 . \tag{11.77}$$

For this scalar product holds

$$\phi^\alpha \chi_\alpha = -\chi^\alpha \phi_\alpha . \tag{11.78}$$

The transformation behaviour of ϕ_α according to (11.63, 11.73, 11.75) is given by

$$\phi'_\alpha = \epsilon_{\alpha\beta} \, a^\beta{}_\gamma \, \epsilon^{\gamma\delta} \, \phi_\delta \tag{11.79}$$

as can be readily verified.

Proof that $\phi^\alpha \chi_\alpha$ is Lorentz invariant

We want to verify now that the scalar product (11.77) is Lorentz invariant. In the transformed frame holds

$$\phi'^\alpha \chi'_\alpha = a^\alpha{}_\beta \, \epsilon_{\alpha\gamma} a^\gamma{}_\delta \epsilon^{\delta\kappa} \, \phi^\beta \chi_\kappa . \tag{11.80}$$

One can write in matrix notation

$$a^\alpha{}_\beta \, \epsilon_{\alpha\gamma} a^\gamma{}_\delta \epsilon^{\delta\kappa} = \left[\left(\epsilon \, a \, \epsilon^{-1} \right)^T \, a \right]_{\kappa\beta} . \tag{11.81}$$

Using (11.2) and (11.14) one can write

$$\epsilon\, a\, \epsilon^{-1} \;=\; \epsilon\, e^{\frac{1}{2}\vec{z}\cdot\vec{\sigma}}\epsilon^{-1} \;=\; e^{\frac{1}{2}\vec{z}\cdot\epsilon\vec{\sigma}\epsilon^{-1}} \;=\; e^{-\frac{1}{2}\vec{z}\cdot\vec{\sigma}^{\,*}} \tag{11.82}$$

and with $f(A)^T = f(A^T)$ for polynomial $f(A)$

$$\left(\epsilon\, a\, \epsilon^{-1}\right)^T \;=\; e^{-\frac{1}{2}\vec{z}\cdot(\vec{\sigma}^{\,*})^T} \;=\; e^{-\frac{1}{2}\vec{z}\cdot\vec{\sigma}} \;=\; a^{-1} \tag{11.83}$$

Here we have employed the hermitian property of $\vec{\sigma}$, i.e., $\vec{\sigma}^{\dagger} = (\vec{\sigma}^{\,*})^T = \vec{\sigma}$. Insertion of (11.83) into (11.81) yields

$$a^{\alpha}{}_{\beta}\, \epsilon_{\alpha\gamma} a^{\gamma}{}_{\delta}\epsilon^{\delta\kappa} \;=\; \left[\left(\epsilon\, a\, \epsilon^{-1}\right)^T a\right]_{\kappa\beta} \;=\; \left[a^{-1}a\right]_{\kappa\beta} \;=\; \delta_{\kappa\beta} \tag{11.84}$$

and, hence, from (11.80)

$$\phi'^{\alpha}\chi'_{\alpha} \;=\; \phi^{\beta}\chi_{\beta}\;. \tag{11.85}$$

The complex conjugate spinors

We consider now the *conjugate complex spinors*

$$(\phi^{\alpha})^* \;=\; \begin{pmatrix} (\phi^1)^* \\ (\phi^2)^* \end{pmatrix}\;. \tag{11.86}$$

A concise notation of the conjugate complex spinors is provided by

$$(\phi^{\alpha})^* \;=\; \phi^{\dot{\alpha}} \;=\; \begin{pmatrix} \phi^{\dot{1}} \\ \phi^{\dot{2}} \end{pmatrix} \tag{11.87}$$

which we will employ from now on. Obviously, it holds $\phi^{\dot{k}} = (\phi^k)^*, k = 1, 2$. The transformation behaviour of $\phi^{\dot{\alpha}}$ is

$$\phi'^{\dot{\alpha}} \;=\; (a^{\alpha}{}_{\beta})^*\, \phi^{\dot{\beta}} \tag{11.88}$$

which one verifies taking the conjugate complex of (11.63). As discussed above, $a^*(\vec{z})$ provides a representation of the Lorentz group which is distinct from that provided by $a(\vec{z})$. Hence, the conjugate complex spinors $\phi^{\dot{\alpha}}$ need to be considered separately from the spinors ϕ^{α}. We denote

$$(a^{\alpha}{}_{\beta})^* \;=\; a^{\dot{\alpha}}{}_{\dot{\beta}} \tag{11.89}$$

such that (11.88) reads

$$\phi'^{\dot{\alpha}} \;=\; a^{\dot{\alpha}}{}_{\dot{\beta}}\phi^{\dot{\beta}} \tag{11.90}$$

extending the summation convention to 'dotted' indices.
We also define covariant versions of $\phi^{\dot{\alpha}}$

$$\phi_{\dot{\alpha}} \;=\; (\phi_{\alpha})^*\;. \tag{11.91}$$

The relationship between contravariant and covariant conjugate complex spinors can be expressed in analogy to (11.73, 11.76)

$$\phi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\,\phi^{\dot\beta} \tag{11.92}$$

$$\phi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\,\phi_{\dot\beta} \tag{11.93}$$

$$\epsilon_{\dot\alpha\dot\beta} = \epsilon_{\alpha\beta} \tag{11.94}$$

$$\epsilon^{\dot\alpha\dot\beta} = \epsilon^{\alpha\beta} \tag{11.95}$$

where $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ are the real matrices defined in (11.75, 11.75). For the spinors $\phi^{\dot\alpha}$ and $\chi_{\dot\alpha}$ thus defined holds that the scalar product

$$\phi^{\dot\alpha}\chi_{\dot\alpha} = \phi^{\dot1}\chi_{\dot1} + \phi^{\dot2}\chi_{\dot2} \tag{11.96}$$

is Lorentz invariant, a property which is rather evident.
The transformation behaviour of $\phi_{\dot\alpha}$ is

$$\phi'_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}a^{\dot\beta}{}_{\dot\gamma}\epsilon^{\dot\gamma\dot\delta}\phi_{\dot\delta}\,. \tag{11.97}$$

The transformation, in matrix notation, is governed by the operator $\epsilon\, a^*(\vec z)\,\epsilon^{-1}$ which arises in the Lorentz transformation (11.18) of the bispinor wave function $\tilde\Psi$, $\epsilon\, a^*(\vec z)\,\epsilon^{-1}$ accounting for the transformation behaviour of the third and fourth spinor component of $\tilde\Psi$. A comparision of (11.18) and (11.97) implies then that $\phi_{\dot\alpha}$ transforms like $\tilde\Psi_3, \tilde\Psi_4$, i.e., one can state

$\phi_{\dot1}$ transforms under rotations ($\vec z = i\vec\vartheta$) like a spin$-\frac12$ state $|\frac12,+\frac12\rangle$
$\phi_{\dot2}$ transforms under rotations ($\vec z = i\vec\vartheta$) like a spin$-\frac12$ state $|\frac12,-\frac12\rangle$.

The transformation behaviour of $\tilde\Psi$ (note that we do not include presently the space-time dependence of the wave function)

$$\tilde\Psi' = \tilde{\mathcal S}(\vec z)\,\tilde\Psi = \begin{pmatrix} a(\vec z)\binom{\tilde\Psi_1}{\tilde\Psi_2} \\ \epsilon a^*(\vec z)\epsilon^{-1}\binom{\tilde\Psi_3}{\tilde\Psi_4}\end{pmatrix} \tag{11.98}$$

obviously implies that the solution of the Dirac equation in the chiral representation can be written in spinor form

$$\begin{pmatrix}\tilde\Psi_1(x^\mu)\\ \tilde\Psi_2(x^\mu)\\ \tilde\Psi_3(x^\mu)\\ \tilde\Psi_4(x^\mu)\end{pmatrix} = \begin{pmatrix}\phi^1(x^\mu)\\ \phi^2(x^\mu)\\ \chi_{\dot1}(x^\mu)\\ \chi_{\dot2}(x^\mu)\end{pmatrix} = \begin{pmatrix}\phi^\alpha(x^\mu)\\ \chi_{\dot\beta}(x^\mu)\end{pmatrix}\,. \tag{11.99}$$

11.4 Spinor Tensors

We generalize now our definition of spinors ϕ^α to tensors. A tensor

$$t^{\alpha_1\alpha_2\cdots\alpha_k\dot\beta_1\dot\beta_2\cdots\dot\beta_\ell} \tag{11.100}$$

is a quantity which under Lorentz transformations behaves as

$$t'^{\alpha_1\alpha_2\cdots\alpha_k\dot\beta_1\dot\beta_2\cdots\dot\beta_\ell} \;=\; \prod_{m=1}^{k} a^{\alpha_m}{}_{\gamma_m} \prod_{n=1}^{\ell} a^{\dot\beta_n}{}_{\dot\delta_n}\, t^{\gamma_1\cdots\gamma_k\dot\delta_1\cdots\dot\delta_\ell}\,. \tag{11.101}$$

An example is the tensor $t^{\alpha\dot\beta}$ which will play an important role in the spinor presentation of the Dirac equation. This tensor transforms according to

$$t'^{\alpha\dot\beta} \;=\; a^{\alpha}{}_{\gamma} a^{\dot\beta}{}_{\dot\delta}\, t^{\gamma\dot\delta} \tag{11.102}$$

This reads in matrix notation, using conventional matrix indices j, k, ℓ, m,

$$t'_{jk} \;=\; \left(a\,t\,a^\dagger\right)_{jk} \;=\; \sum_{\ell,m} a_{j\ell} a^*_{km} t_{\ell m}\,. \tag{11.103}$$

Similarly, the transformation bevaviour of a tensor $t^{\alpha\beta}$ reads in spinor and matrix notation

$$t'^{\alpha\beta} \;=\; a^{\alpha}{}_{\gamma}\, a^{\beta}{}_{\delta}\, t^{\gamma\delta}\,, \qquad t'_{jk} \;=\; \left(a\,t\,a^T\right)_{jk} \;=\; \sum_{\ell,m} a_{j\ell} a_{km} t_{\ell m} \tag{11.104}$$

Indices on tensors can also be lowered employing the formula

$$t_{\alpha}{}^{\dot\beta} \;=\; \epsilon_{\alpha\gamma} t^{\gamma\dot\beta} \tag{11.105}$$

and generalizations thereof.
An example of a tensor is $\epsilon^{\alpha\beta}$ and $\epsilon_{\alpha\beta}$. This tensor is actually invariant under Lorentz transformations, i.e., it holds

$$\epsilon'^{\alpha\beta} \;=\; \epsilon^{\alpha\beta}\,, \qquad \epsilon'_{\alpha\beta} \;=\; \epsilon_{\alpha\beta} \tag{11.106}$$

Exercise 11.4.1: Prove equation (11.106).

The 4–vector A^μ in spinor form

We want to provide now the spinor form of the 4-vector A^μ, i.e., we want to express A^μ through a spinor tensor. This task implies that we seek a tensor, the elements of which are linear functions of A^μ. An obvious candidate is [cf. (11.25] $M(A^\mu) = \sigma_\mu A^\mu$. We had demonstrated that $M(A^\mu)$ transforms according to

$$M' \;=\; M(L^{\mu}{}_{\nu}A^\nu) \;=\; a\,M(A^\mu)\,a^\dagger \tag{11.107}$$

which reads in spinor notation [cf. (11.102, 11.103)

$$A'^{\alpha\dot\beta} \;=\; a^{\alpha}{}_{\gamma} a^{\dot\beta}{}_{\dot\delta} A^{\gamma\dot\delta}\,. \tag{11.108}$$

Obviously, this transformation behaviour is in harmony with the tensor notation adopted, i.e., with contravariant indices $\alpha\dot\beta$. According to (11.25) the tensor is explicitly

$$A^{\alpha\dot\beta} \;=\; \begin{pmatrix} A^{1\dot1} & A^{1\dot2} \\ A^{2\dot1} & A^{2\dot2} \end{pmatrix} \;=\; \begin{pmatrix} A^0 + A^3 & A^1 - iA^2 \\ A^1 + iA^2 & A^0 - A^3 \end{pmatrix}\,. \tag{11.109}$$

One can express $A^{\alpha\dot\beta}$ also through A_μ

$$A^{\alpha\dot\beta} \;=\; \begin{pmatrix} A_0 \,-\, A_3 & -A_1 \,+\, iA_2 \\ -A_1 \,-\, iA_2 & A_0 \,+\, A_3 \end{pmatrix} . \tag{11.110}$$

The 4-vectors A^μ, A_μ can also be associated with tensors

$$A_{\alpha\dot\beta} \;=\; \epsilon_{\alpha\gamma}\epsilon_{\dot\beta\dot\delta}A^{\gamma\dot\delta} . \tag{11.111}$$

This tensor reads in matrix notation

$$\begin{aligned} \begin{pmatrix} A_{1\dot1} & A_{1\dot2} \\ A_{2\dot1} & A_{2\dot2} \end{pmatrix} \;&=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A^{1\dot1} & A^{1\dot2} \\ A^{2\dot1} & A^{2\dot2} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &=\; \begin{pmatrix} A^{2\dot2} & -A^{2\dot1} \\ -A^{1\dot2} & A^{1\dot1} \end{pmatrix} . \end{aligned} \tag{11.112}$$

Hence, employing (11.109, 11.110) one obtains

$$A_{\alpha\dot\beta} \;=\; \begin{pmatrix} A^0 \,-\, A^3 & -A^1 \,-\, iA^2 \\ -A^1 \,+\, iA^2 & A^0 \,+\, A^3 \end{pmatrix} \tag{11.113}$$

$$A_{\alpha\dot\beta} \;=\; \begin{pmatrix} A_0 \,+\, A_3 & A_1 \,+\, iA_2 \\ A_1 \,-\, iA_2 & A_0 \,-\, A_3 \end{pmatrix} . \tag{11.114}$$

We finally note that the 4–vector scalar product $A^\mu B_\mu$ reads in spinor notation

$$A^\mu B_\mu \;=\; \frac{1}{2} A^{\alpha\dot\beta} B_{\alpha\dot\beta} . \tag{11.115}$$

Exercise 11.4.2: Prove that (11.115) is correct.

∂_μ in spinor notation

The relationship between 4–vectors A^μ, A_μ and tensors $t^{\alpha\dot\beta}$ can be applied to the partial differential operator ∂_μ. Using (11.110) one can state

$$\partial^{\alpha\dot\beta} \;=\; \begin{pmatrix} \partial_0 \,-\, \partial_3 & -\partial_1 \,+\, i\partial_2 \\ -\partial_1 \,-\, i\partial_2 & \partial_0 \,+\, \partial_3 \end{pmatrix} . \tag{11.116}$$

Similarly, (11.114) yields

$$\partial_{\alpha\dot\beta} \;=\; \begin{pmatrix} \partial_0 \,+\, \partial_3 & \partial_1 \,+\, i\partial_2 \\ \partial_1 \,-\, i\partial_2 & \partial_0 \,-\, \partial_3 \end{pmatrix} . \tag{11.117}$$

σ_μ in Tensor Notation

We want to develop now the tensor notation for σ_μ (11.26) and its contravariant analogue σ^μ

$$\begin{aligned} \sigma_\mu &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \\ \sigma^\mu &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right) \end{aligned} \tag{11.118}$$

For this purpose we consider first the transformation behaviour of σ^μ and σ_μ. We will obtain the transformation behaviour of σ^μ, σ_μ building on the known transformation behaviour of $\tilde{\gamma}^\mu$. This is possible since $\tilde{\gamma}^\mu$ can be expressed through σ^μ, σ_μ. Comparision of (10.229) and (11.118) yields

$$\tilde{\gamma}^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma_\mu & 0 \end{pmatrix} . \tag{11.119}$$

Using (11.15) one can write

$$\sigma_\mu = \epsilon \, (\sigma^\mu)^* \, \epsilon^{-1} \tag{11.120}$$

and, hence,

$$\tilde{\gamma}^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \epsilon \, (\sigma^\mu)^* \, \epsilon^{-1} & 0 \end{pmatrix} . \tag{11.121}$$

One expects then that the transformation properties of σ^μ should follow from the transformation properties established already for γ^μ [c.f. (10.243)]. Note that (10.243) holds independently of the representation chosen, i.e., holds also in the chiral representation.
To obtain the transformation properties of σ^μ we employ then (10.243) in the chiral representation expressing $\mathcal{S}(L^\eta{}_\xi)$ by (11.18) and γ^μ by (11.121). Equation (10.243) reads then

$$\begin{pmatrix} a & 0 \\ 0 & \epsilon a^* \epsilon^{-1} \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \epsilon (\sigma^\mu)^* \epsilon^{-1} & 0 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & \epsilon a^* \epsilon^{-1} \end{pmatrix} L^\nu{}_\mu = \begin{pmatrix} 0 & \sigma^\nu \\ \epsilon (\sigma^\nu)^* \epsilon^{-1} & 0 \end{pmatrix} . \tag{11.122}$$

The l.h.s. of this equation is

$$\begin{pmatrix} 0 & a \sigma^\mu \epsilon^{-1} (a^*)^{-1} \epsilon \\ \epsilon a^* (\sigma^\mu)^* \epsilon^{-1} a^{-1} & 0 \end{pmatrix} L^\nu{}_\mu \tag{11.123}$$

and, hence, one can conclude

$$a \sigma^\mu \epsilon^{-1} (a^*)^{-1} \epsilon \, L^\nu{}_\mu = \sigma^\nu \tag{11.124}$$

$$\epsilon a^* (\sigma^\mu)^* \epsilon^{-1} a^{-1} \, L^\nu{}_\mu = \epsilon (\sigma^\nu)^* \epsilon^{-1} . \tag{11.125}$$

Equation (11.125) is equivalent to

$$a^* (\sigma^\mu)^* \epsilon^{-1} a^{-1} \epsilon \, L^\nu{}_\mu = (\sigma^\nu)^* \tag{11.126}$$

which is the complex conjugate of (11.124), i.e., (11.125) is equivalent to (11.124). Hence, (11.124) constitutes the essential transformation property of σ^μ and will be considered further.

One can rewrite (11.124) using (11.16, 11.2)

$$\epsilon^{-1}(a^*)^{-1}\epsilon \;=\; \epsilon^{-1}e^{-\frac{1}{2}\vec{z}^*\cdot\vec{\sigma}^*}\epsilon \;=\; e^{-\frac{1}{2}\vec{z}^*\cdot\epsilon^{-1}\vec{\sigma}^*\epsilon} \;=\; e^{\frac{1}{2}\vec{z}^*\cdot\vec{\sigma}} \,. \tag{11.127}$$

Exploiting the hermitian property of $\vec{\sigma}$, e.g., $(\vec{\sigma}^*)^T = \vec{\sigma}$ yields using (11.2)

$$\epsilon^{-1}(a^*)^{-1}\epsilon \;=\; e^{\frac{1}{2}\vec{z}^*\cdot(\vec{\sigma}^*)^T} \;=\; \left[e^{\frac{1}{2}\vec{z}^*\cdot\vec{\sigma}^*}\right]^T \;=\; [a^*]^T \,. \tag{11.128}$$

One can express, therefore, equation (11.124)

$$a\,\sigma^\mu\;[a^*]^T\;L^\nu{}_\mu \;=\; \sigma^\nu \,. \tag{11.129}$$

We want to demonstrate now that the expression $a\sigma^\mu[a^*]^T$ is to be interpreted as the transform of σ^μ under Lorentz transformations. In fact, under rotations the Pauli matrices transform like $(j = 1,2,3)$

$$\sigma_j \;\longrightarrow\; a(i\vec{\vartheta})\,\sigma_j\left(a(i\vec{\vartheta})\right)^\dagger \;=\; a(i\vec{\vartheta})\,\sigma_j\left(a^*(i\vec{\vartheta})\right)^T \,. \tag{11.130}$$

We argue in analogy to the logic applied in going from (11.107) to (11.108) that the same transformation behaviour applies then for general Lorentz transformations, i.e., transformations (11.2, 11.4) with $\vec{w} \neq 0$. One can, hence, state that σ^μ in a new reference frame is

$$\sigma'^\mu \;=\; a\,\sigma^\mu\,a^\dagger \tag{11.131}$$

where a is given by (11.2, 11.4). This transformation behaviour, according to (11.102, 11.103) identifies σ^μ as a tensor of type $t^{\alpha\dot{\beta}}$. It holds according to (11.118)

$$\begin{pmatrix} (\sigma^\mu)^{1\dot{1}} & (\sigma^\mu)^{1\dot{2}} \\ (\sigma^\mu)^{2\dot{1}} & (\sigma^\mu)^{2\dot{2}} \end{pmatrix} = \left(\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mu=0}, \underbrace{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}_{\mu=1}, \underbrace{\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}}_{\mu=2}, \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mu=3}, \right) \tag{11.132}$$

and

$$\begin{pmatrix} (\sigma_\mu)^{1\dot{1}} & (\sigma_\mu)^{1\dot{2}} \\ (\sigma^\mu)_{2\dot{1}} & (\sigma^\mu)_{2\dot{2}} \end{pmatrix} = \left(\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mu=0}, \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\mu=1}, \underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\mu=2}, \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\mu=3}, \right) \,. \tag{11.133}$$

Combining (11.129, 11.131) one can express the transformation behaviour of σ^μ in the succinct form

$$L^\nu{}_\mu\,\sigma'^\mu \;=\; \sigma^\nu \,. \tag{11.134}$$

Inverting contravariant and covariant indices one can also state

$$L_\nu{}^\mu \, \sigma'_\mu \; = \; \sigma_\nu \; . \tag{11.135}$$

This is the property surmised already above [cf. (11.29)]. We can summarize that σ^μ and σ_μ transform like a 4–vector, however, the transformation is *inverse* to that of ordinary 4–vectors. Each of the $4 \times 4 = 16$ matrix elements in (11.132) and (11.133) is characterized through a 4-vector index $\mu, \mu = 0, 1, 2, 3$ as well as through two spinor indices $\alpha\dot{\beta}$. We want to express now σ^μ and σ_μ also with respect to the 4–vector index μ in spinor form employing (11.114). This yields

$$\sigma_{\alpha\dot{\beta}} \; = \; \begin{pmatrix} \sigma_0 + \sigma_3 & \sigma_1 + i\sigma_2 \\ \sigma_1 - i\sigma_2 & \sigma_0 - \sigma_3 \end{pmatrix} \; = \; 2 \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \tag{11.136}$$

where on the rhs. the submatrices correspond to $t^{\alpha\dot{\beta}}$ spinors. We can, in fact, state

$$\left(\sigma_{\alpha\dot{\beta}}\right)^{\gamma\dot{\delta}} \; = \; \begin{pmatrix} \begin{pmatrix} (\sigma_{1\dot{1}})^{1\dot{1}} & (\sigma_{1\dot{1}})^{1\dot{2}} \\ (\sigma_{1\dot{1}})^{2\dot{1}} & (\sigma_{1\dot{1}})^{2\dot{2}} \end{pmatrix} & \begin{pmatrix} (\sigma_{1\dot{2}})^{1\dot{1}} & (\sigma_{1\dot{2}})^{1\dot{2}} \\ (\sigma_{1\dot{2}})^{2\dot{1}} & (\sigma_{1\dot{2}})^{2\dot{2}} \end{pmatrix} \\ \begin{pmatrix} (\sigma_{2\dot{1}})^{1\dot{1}} & (\sigma_{2\dot{1}})^{1\dot{2}} \\ (\sigma_{2\dot{1}})^{2\dot{1}} & (\sigma_{2\dot{1}})^{2\dot{2}} \end{pmatrix} & \begin{pmatrix} (\sigma_{2\dot{2}})^{1\dot{1}} & (\sigma_{2\dot{2}})^{1\dot{2}} \\ (\sigma_{2\dot{2}})^{2\dot{1}} & (\sigma_{2\dot{2}})^{2\dot{2}} \end{pmatrix} \end{pmatrix} \; . \tag{11.137}$$

Equating this with the r.h.s. of (11.136) results in the succinct expression

$$\frac{1}{2} \left(\sigma_{\alpha\dot{\beta}}\right)^{\gamma\dot{\delta}} \; = \; \delta_{\alpha\gamma}\delta_{\dot{\beta}\dot{\delta}} \; . \tag{11.138}$$

Note that all elements of $\sigma_{\alpha\dot{\beta}}$ are real and that there are only four non-vanishing elements.
In (11.138) the 'inner' covariant spinor indices, i.e., $\alpha, \dot{\beta}$, account for the 4–vector index μ, whereas the 'outer' contravariant spinor indices, i.e., $\gamma, \dot{\delta}$, account for the elements of the individual Pauli matrices. We will now consider the representation of σ^μ, σ_μ in which the contravariant indices are moved 'inside', i.e., account for the 4-vector μ, and the covariant indices are moved outside. The desired change of representation $(\sigma_\mu)^{\alpha\dot{\beta}} \longrightarrow (\sigma_\mu)_{\alpha\dot{\beta}}$ corresponds to a transformation of the basis of spin states

$$\begin{pmatrix} f \\ g \end{pmatrix} \longrightarrow \begin{pmatrix} g \\ -f \end{pmatrix} \; = \; \epsilon \begin{pmatrix} f \\ g \end{pmatrix} \tag{11.139}$$

and, hence, corresponds to the transformation

$$\begin{pmatrix} (\sigma_\mu)_{1\dot{1}} & (\sigma_\mu)_{1\dot{2}} \\ (\sigma_\mu)_{2\dot{1}} & (\sigma_\mu)_{2\dot{2}} \end{pmatrix} \; = \; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} (\sigma_\mu)^{1\dot{1}} & (\sigma_\mu)^{1\dot{2}} \\ (\sigma_\mu)^{2\dot{1}} & (\sigma_\mu)^{2\dot{2}} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{11.140}$$

where we employed the expressions (11.12) for ϵ and ϵ^{-1}. Using (11.110) to express $\sigma^{\alpha\dot{\beta}}$ in terms of σ_μ yields together with (5.224)

$$\sigma^{\alpha\dot{\beta}} \; = \; \begin{pmatrix} \sigma_0 - \sigma_3 & \sigma_1 + i\sigma_2 \\ -\sigma_1 - i\sigma_2 & \sigma_0 + \sigma_3 \end{pmatrix} \; = \; 2 \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \tag{11.141}$$

and, employing then transformation (11.140) to transform each of the four submatrices, which in (11.137) are in a basis $(\cdots)^{\alpha\dot\beta}$ to a basis $(\cdots)_{\alpha\dot\beta}$ results in

$$\begin{aligned}
\left(\sigma^{\alpha\dot\beta}\right)_{\gamma\dot\delta} &= \begin{pmatrix} \begin{pmatrix} (\sigma^{1\dot 1})_{1\dot 1} & (\sigma^{1\dot 1})_{1\dot 2} \\ (\sigma^{1\dot 1})_{2\dot 1} & (\sigma^{1\dot 1})_{2\dot 2} \end{pmatrix} & \begin{pmatrix} (\sigma^{1\dot 2})_{1\dot 1} & (\sigma^{1\dot 2})_{1\dot 2} \\ (\sigma^{1\dot 2})_{2\dot 1} & (\sigma^{1\dot 2})_{2\dot 2} \end{pmatrix} \\ \begin{pmatrix} (\sigma^{2\dot 1})_{1\dot 1} & (\sigma^{2\dot 1})_{1\dot 2} \\ (\sigma^{2\dot 1})_{2\dot 1} & (\sigma^{2\dot 1})_{2\dot 2} \end{pmatrix} & \begin{pmatrix} (\sigma^{2\dot 2})_{1\dot 1} & (\sigma^{2\dot 2})_{1\dot 2} \\ (\sigma^{2\dot 2})_{2\dot 1} & (\sigma^{2\dot 2})_{2\dot 2} \end{pmatrix} \end{pmatrix} \\
&= 2 \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} . \qquad (11.142)
\end{aligned}$$

This can be expressed

$$\frac{1}{2}\left(\sigma^{\alpha\dot\beta}\right)_{\gamma\dot\delta} = \delta_{\alpha\gamma}\delta_{\dot\beta\dot\delta} . \qquad (11.143)$$

Combined with (11.138) one can conclude that the following property holds

$$\frac{1}{2}\left(\sigma_{\alpha\dot\beta}\right)^{\gamma\dot\delta} = \frac{1}{2}\left(\sigma^{\alpha\dot\beta}\right)_{\gamma\dot\delta} = \delta_{\alpha\gamma}\delta_{\dot\beta\dot\delta} . \qquad (11.144)$$

The Dirac Matrices γ^μ in spinor notation

We want to express now the Dirac matrices $\tilde\gamma^\mu$ in spinor form. For this purpose we start from the expression (11.121) of $\tilde\gamma^\mu$. This expression implies that the element of $\tilde\gamma^\mu$ given by $\epsilon\,\sigma^\mu\epsilon^{-1}$ is in the basis $|_{\alpha\dot\beta}$ whereas the element of $\tilde\gamma^\mu$ given by σ^μ is in the basis $|^{\alpha\dot\beta}$. Accordingly, we write

$$\tilde\gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)^{\alpha\dot\beta} \\ ((\sigma^\mu)_{\alpha\dot\beta})^* & 0 \end{pmatrix} . \qquad (11.145)$$

Let A_μ be a covariant 4–vector. One can write then the scalar product using (11.115)

$$\begin{aligned}
\tilde\gamma^\mu A_\mu &= \begin{pmatrix} 0 & (\sigma^\mu)^{\alpha\dot\beta} A_\mu \\ ((\sigma^\mu)_{\alpha\dot\beta})^* A_\mu & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{1}{2}(\sigma_{\gamma\dot\delta})^{\alpha\dot\beta} A^{\gamma\dot\delta} \\ \frac{1}{2}((\sigma^{\gamma\dot\delta})_{\alpha\dot\beta})^* A_{\gamma\dot\delta} & 0 \end{pmatrix} . \qquad (11.146)
\end{aligned}$$

Exploiting the property (11.144) results in the simple relationship

$$\tilde\gamma^\mu A_\mu = \begin{pmatrix} 0 & A^{\alpha\dot\beta} \\ A_{\alpha\dot\beta} & 0 \end{pmatrix} . \qquad (11.147)$$

11.5 Lorentz Invariant Field Equations in Spinor Form

Dirac Equation

(11.147) allows us to rewrite the Dirac equation in the chiral representation (10.226)

$$i\,\gamma^\mu \partial_\mu \tilde{\Psi}(x^\mu) \;=\; \begin{pmatrix} 0 & i\,\partial^{\alpha\dot{\beta}} \\ i\,\partial_{\alpha\dot{\beta}} & 0 \end{pmatrix} \tilde{\Psi}(x^\mu) \;=\; m\,\tilde{\Psi}(x^\mu)\;. \tag{11.148}$$

Employing $\tilde{\Psi}(x^\mu)$ in the form (11.99) yields the Dirac equation in spinor form

$$i\,\partial^{\alpha\dot{\beta}}\,\chi_{\dot{\beta}} \;=\; m\,\phi^\alpha \tag{11.149}$$
$$i\,\partial_{\alpha\dot{\beta}}\,\phi^\alpha \;=\; m\,\chi_{\dot{\beta}}\;. \tag{11.150}$$

The simplicity of this equation is striking.