8.323: Relativistic Quantum Field Theory I

INFORMAL NOTES
DISTRIBUTIONS AND THE FOURIER TRANSFORM

Basic idea:

In QFT it is common to encounter integrals that are not well-defined. Peskin and Schroeder, for example, give the following formula (p. 27, after Eq. (2.51)) for the two point function $\langle 0 \,|\phi(x)\phi(y)|\, 0\rangle$ for spacelike separations $(x-y)^2 = -r^2$:

$$D(r) = \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^{\infty} \mathrm{d}p \, \frac{p e^{ipr}}{\sqrt{p^2+m^2}} \ .$$

If this integral is defined in the usual way as

$$\lim_{\Lambda\to\infty} \int_{-\Lambda}^{\Lambda} \mathrm{d}p \, \frac{p e^{ipr}}{\sqrt{p^2+m^2}} \ ,$$

then it does not exist. The integral can be defined by putting in a convergence factor $e^{-\epsilon|p|}$:

$$\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \mathrm{d}p \, \frac{p e^{ipr} e^{-\epsilon|p|}}{\sqrt{p^2+m^2}} \ .$$

But how does one know whether a different convergence factor would get the same result? One way to resolve these issues is to treat the ambiguous quantity as a distribution, rather than a function. All tempered distributions (to be defined below) have Fourier transforms, which are also tempered distributions. Furthermore, we can show that the ϵ-prescription used above is equivalent to the tempered-distribution definition of the Fourier transform.

Distribution:

A distribution is a linear mapping from a space of test functions to real or complex numbers. (An operator-valued distribution maps test functions into operators.)

Test Functions:

The space of test functions $\{\varphi(t)\}$ determines what type of distribution one is discussing. The test functions for tempered distributions belong to "Schwartz space," the space of functions which are infinitely differentiable, and the function and each of its derivatives fall off faster than any power for large t. The Gaussian is a good example of a Schwartz function. Any function in Schwartz space has a Fourier transform in Schwartz space. (The Fourier transform of a Gaussian is a Gaussian.)

Functions as Distributions:

Distributions are sometimes called generalized functions, which suggests that a function is also a distribution. This is not quite true, but a wide range of functions can also be thought of as distributions. Given any function $f(t)$ which is piecewise continuous and bounded by some power of t for large t, one can define a corresponding distribution T_f by

$$T_f[\varphi] \equiv \int_{-\infty}^{\infty} \mathrm{d}t\, f(t)\varphi(t)\ .$$

Since $\varphi(t)$ falls off faster than any power, this integral will converge. Note that because the class of $\varphi(t)$'s is very restricted, the class of possible $f(t)$'s is very large.

Fourier Transform:

For any function $f(t)$ which is integrable, meaning that

$$\int_{-\infty}^{\infty} \mathrm{d}t\, |f(t)|$$

converges, define

$$\tilde{f}(\omega) \equiv \int_{-\infty}^{\infty} \mathrm{d}t\, e^{-i\omega t} f(t)\ .$$

Fourier Transform of a Distribution:

To motivate the definition, suppose $f(t)$ is integrable, and consider

$$\begin{aligned} \tilde{T}_f[\varphi] &= \int_{-\infty}^{\infty} \tilde{f}(\omega)\varphi(\omega)\, d\omega \\ &= \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} e^{-i\omega t}\, f(t)\, \varphi(\omega)\, d\omega \\ &= \int_{-\infty}^{\infty} \mathrm{d}t\, f(t)\, \tilde{\varphi}(t) \\ &= T_f[\tilde{\varphi}]\ . \end{aligned}$$

Note that these integrals are absolutely convergent, so there is no problem about interchanging the order of integration. So, for any distribution T, define its Fourier transform by

$$\boxed{\tilde{T}[\varphi] \equiv T[\tilde{\varphi}]\ .}$$

Note that any function $f(t)$ which is piecewise continuous and bounded by some power of t for large t can define a distribution, and can therefore be Fourier transformed as a distribution.

Relation to ϵ convergence factor:

Suppose $f(t)$ is not integrable, and so does not have a Fourier transform. Suppose, however, that there exists a continuous sequence of "regulated functions" $f_\epsilon(t)$ which are integrable for $\epsilon > 0$, which satisfy

$$|f_\epsilon(t)| < |f(t)| \ ,$$

and which for each t satisfy

$$\lim_{\epsilon\to 0} f_\epsilon(t) = f(t) \ .$$

Example: $f_\epsilon(t) = f(t)e^{-\epsilon|t|}$. Note that the regulator that we used for the two-point function at spacelike separations has this property. To show: if we Fourier transform $f_\epsilon(t)$ and take the limit $\epsilon \to 0$ at the end, it is the same as the distribution-theory definition of the Fourier transform.

Proof:

The distribution-theory definition of the Fourier transform is

$$\begin{aligned} \tilde{T}_f[\varphi] &\equiv T_f[\tilde{\varphi}] \\ &= \boxed{\int_{-\infty}^{\infty} \mathrm{d}t\, f(t)\, \tilde{\varphi}(t) \ .} \end{aligned}$$

The ϵ prescription is to use

$$T^*_f[\varphi] \equiv \lim_{\epsilon\to 0} T_{\tilde{f}_\epsilon}[\varphi] \ .$$

We need to show these are equivalent. Use

$$\begin{aligned} T^*_f[\varphi] &= \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \mathrm{d}\omega\, \tilde{f}_\epsilon(\omega)\varphi(\omega) \\ &= \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \mathrm{d}\omega \int_{-\infty}^{\infty} \mathrm{d}t\, e^{-i\omega t}\, f_\epsilon(t)\varphi(\omega) \\ &= \boxed{\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \mathrm{d}t\, f_\epsilon(t)\tilde{\varphi}(t) \ .} \end{aligned}$$

If we can take the limit inside the integral, we are done!

Last step is proven with Lebesgue's Dominated Convergence Theorem: If $h_\epsilon(t)$ is a sequence of functions for which

$$\lim_{\epsilon\to 0} h_\epsilon(t) = h(t) \qquad \text{for all } t,$$

and if there exists a function $g(t)$ for which

$$\int \mathrm{d}t\, g(t)$$

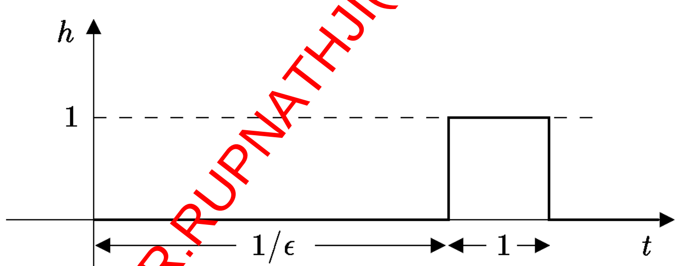
converges, and for which

$$g(t) \geq |h_\epsilon(t)| \qquad \text{for all } t \text{ and all } \epsilon,$$

then

$$\lim_{\epsilon\to 0} \int \mathrm{d}t\, h_\epsilon(t) = \int \mathrm{d}t\, h(t) \ .$$

Note, by the way, that the existence of the integrable bounding function $g(x)$ is absolutely necessary. A simple example of a function $h_\epsilon(t)$ for which one CANNOT bring the limit through the integral sign would be a function that looks something like:



Analytically, this function can be written as

$$h_\epsilon(t) = \begin{cases} 1 & \text{if } \frac{1}{\epsilon} < t < \frac{1}{\epsilon} + 1 \\ 0 & \text{otherwise} \ . \end{cases}$$

Note that the square well moves infinitely far to the right as $\epsilon \to 0$, so $h_\epsilon(t) \to 0$ for any t. But the integral of the curve is 1 for any ϵ, and hence it is 1 in the limit. The Lebesgue Dominated Convergence theorem excludes functions like this, because any bounding function $g(t)$ must be ≥ 1 for all t, so $g(t)$ cannot be integrable.

The theorem does apply, however, to

$$\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \mathrm{d}t\, f_\epsilon(t) \tilde{\varphi}(t) \ .$$

Take

$$h_\epsilon(t) = f_\epsilon(t)\,\tilde{\varphi}(t)\ ,$$
$$h(t) = f(t)\,\tilde{\varphi}(t)\ ,$$

and

$$g(t) = |f(t)\,\tilde{\varphi}(t)|\ .$$

Bottom Line:

The ϵ prescription used by physicists is equivalent to the unambiguous definition of the Fourier transform in tempered-distribution theory. That is, if the function to be Fourier-transformed $f(t)$ is not integrable, one can proceed as long as one can find an integrable regulator $f_\epsilon(t)$ such that

$$|f_\epsilon(t)| < |f(t)|\ ,$$

and for each t,

$$\lim_{\epsilon \to 0} f_\epsilon(t) = f(t)\ .$$

One can then Fourier transform $f_\epsilon(t)$ instead. In the general case one cannot take the limit $\epsilon \to 0$ immediately, but one must leave ϵ in the expression for the distribution. Only after the distribution is evaluated for a particular test function can the limit $\epsilon \to 0$ be taken. Remember, for example, that we wrote the Fourier transform of the Feynman propagator as

$$\frac{i}{p^2 - m^2 + i\epsilon}\ .$$

With the ϵ in place one can carry out integrals involving the propagator, and then one can take the limit $\epsilon \to 0$ at the end. If one tried to set the ϵ term to zero immediately, then the poles in the propagator would lead to ill-defined integrations.