Quantization of the Free Scalar Field

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— Alan Guth

Alan Guth
Massachusetts Institute of Technology
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QUANTIZATION OF THE FREE SCALAR FIELD

1. The Lagrangian:

$$L = \int d^3x \, \mathscr{L}$$

where

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$
$$= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\nabla_i\phi\,\nabla_i\phi - \frac{1}{2}m^2\phi^2$$

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2. Canonical Quantization:

$$L = L(q_i, \dot{q}_i, t)$$

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

$$H = \sum_i p_i \, \dot{q}_i - L$$

$$[q_i \, , \, p_j] = i\hbar \, \delta_{ij}$$

Schrödinger picture:

$$i\hbar \frac{\partial}{\partial t} \, |\psi(t)\rangle = H \, |\psi(t)\rangle$$

If H is independent of time,

$$|\psi(t)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle \ .$$

Heisenberg picture:

$$\mathcal{O}(t) = e^{iHt/\hbar} \, \mathcal{O} \, e^{-iHt/\hbar} \ ,$$

so that

$$\left\langle \psi \middle| \mathcal{O}(t) \middle| \psi \right\rangle = \left\langle \psi(t) \middle| \mathcal{O} \middle| \psi(t) \right\rangle \ .$$

Relativistic quantum field theory is usually done in the Heisenberg picture: puts space and time on equal footing.

3. Field Quantization by Lattice Approximation:

Motivation:

1) Can use canonical formulation exactly, instead of as a starting point for a "natural" generalization. Note that δ_{ij} might naturally generalize to $\delta^3(\vec{x}\,' - \vec{x})$, but the two expressions do not even have the same units. δ_{ij} is dimensionless, while $\delta^3(\vec{x}\,' - \vec{x})$ has units of $1/L^3$.

2) For interacting theories, the lattice formulation is the easiest way to understand renormalization. When strong coupling is essential, as in QCD (Quantum Chromodynamics) and the strong interactions, the lattice is even the best way to calculate.

Start with discrete lattice, spacing $= a$, and finite volume. Later take limits $a \to 0$ and $V \to \infty$. For the free theory, these limits are trivial.

$$L = \sum_k \mathscr{L}_k \, \Delta V \ ,$$

where

$$\Delta V = a^3$$

and

$$\mathscr{L}_k = \frac{1}{2}\dot{\phi}_k^2 - \frac{1}{2}\nabla_i \phi_k \, \nabla_i \phi_k - \frac{1}{2} m^2 \phi_k^2 \ ,$$

and the lattice derivative is defined by

$$\nabla_i \phi_k \equiv \frac{\phi_{k'(k,i)} - \phi_k}{a} \ ,$$

where $k'(k,i)$ denotes the lattice site that is a distance a in the ith direction from the lattice site k.

Canonical momenta:

$$p_k = \frac{\partial L}{\partial \dot{\phi}_k} = \dot{\phi}_k \, \Delta V \ .$$

Define a canonical momentum density:

$$\pi_k \equiv \frac{p_k}{\Delta V} = \dot{\phi}_k \ .$$

Hamiltonian:

$$H = \left[\sum_k p_k \dot{\phi}_k \right] - L = \sum_k \left[\pi_k \dot{\phi}_k \, - \, \mathcal{L}_k \right] \Delta V \ .$$

Canonical commutation relations:

$$[\phi_{k'} \, , \, \phi_k] = 0 \ , \ [p_{k'} \, , \, p_k] = 0 \ , \quad \text{and} \quad [\phi_{k'} \, , \, p_k] = i\hbar \, \delta_{k'k} \ .$$

In terms of the canonical momentum densities,

$$[\phi_{k'} \, , \, \phi_k] = 0 \ , \ [\pi_{k'} \, , \, \pi_k] = 0 \ , \quad \text{and} \quad [\phi_{k'} \, , \, \pi_k] = \frac{i\hbar \ \delta_{k'k}}{\Delta V} \ .$$

Continuum limit:

$$k \quad \Longrightarrow \quad \vec{x}$$

$$\pi_k \equiv \frac{1}{\Delta V} \frac{\partial L}{\partial \dot{\phi}_k} = \frac{\partial \mathcal{L}_k}{\partial \dot{\phi}_k} \quad \Longrightarrow \quad \pi(\vec{x}\,,t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\vec{x}\,,t)} = \dot{\phi}(\vec{x}\,,t) \ .$$

Hamiltonian becomes

$$H = \sum_k \left[\pi_k \dot{\phi}_k \, - \, \mathcal{L}_k \right] \Delta V \ \Longrightarrow$$

$$H = \int \mathrm{d}^3 x \, \left[\pi \dot{\phi} - \mathcal{L} \right] \ .$$

Canonical commutation relations:

$$\left[\phi(\vec{x}\,',t)\,,\,\phi(\vec{x},t)\right] = 0 \quad \text{and} \quad \left[\pi(\vec{x}\,',t)\,,\,\pi(\vec{x},t)\right] = 0 \ ,$$

$$[\phi_{k'}\,,\,\pi_k] = \frac{i\hbar\,\delta_{k'k}}{\Delta V} \quad \Longrightarrow \quad \left[\phi(\vec{x}\,',t)\,,\,\pi(\vec{x},t)\right] = i\hbar\,\delta(\vec{x}-\vec{x}\,') \ ,$$

since

$$\sum_{k\in\mathcal{R}} \frac{\delta_{k'k}}{\Delta V}\,\Delta V = \begin{cases} 1 & \text{if } k'\in\mathcal{R} \\ 0 & \text{otherwise} \end{cases} \quad \Longrightarrow \int_{\vec{x}\in\mathcal{R}} \mathrm{d}^3x\,\delta^3(\vec{x}\,'-\vec{x})$$

Note: Dirac delta function is defined by

$$\int_{\vec{x}\in\mathcal{R}} \mathrm{d}^3x\, f(\vec{x})\,\delta(\vec{x}-\vec{x}\,') = \begin{cases} f(\vec{x}\,') & \text{if } x'\in\mathcal{R} \\ 0 & \text{otherwise.} \end{cases}$$

4. Review of Simple Harmonic Oscillator:

Choose $m \equiv 1$, so

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 \ ,$$

$$p = \frac{\partial L}{\partial \dot{q}} = \dot{q} \ ,$$

$$H = p\dot{q} - L = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 \ .$$

Canonical commutation relation:

$$[q\,,\,p] = i\hbar \ .$$

Define creation and annihilation operators

$$a = \sqrt{\frac{\omega}{2\hbar}}\,q + \frac{i}{2\hbar\omega}\,p \ , \quad a^\dagger = \sqrt{\frac{\omega}{2\hbar}}\,q - \frac{i}{2\hbar\omega}\,p \ ,$$

so that

$$\left[a\,,\,a^\dagger\right] = 1 \ .$$

The Hamiltonian can be rewritten as

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) ,$$

with eigenstates $|n\rangle$ with eigenvalues

$$E_n = \left(n + \frac{1}{2} \right) \hbar\omega .$$

Then

$$\left[H \, , \, a^\dagger \right] = \hbar\omega \, a^\dagger \ , \quad [H \, , \, a] = -\hbar\omega \, a \, ,$$

which implies that a lowers, and $a^\dagger$ raises, the eigenvalues of H by $\hbar\omega$.

$$a \, |0\rangle = 0 ,$$

Normalized nth excited state:

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(a^\dagger \right)^n |n\rangle ,$$

Solve for q and p:

$$q = \sqrt{\frac{\hbar}{2\omega}} \left(a + a^\dagger \right) , \quad p = -i \sqrt{\frac{\hbar\omega}{2}} \left(a - a^\dagger \right) .$$

In the Heisenberg picture,

$$q(t) = \sqrt{\frac{\hbar}{2\omega}} \left(a e^{-i\omega t} + a^\dagger e^{i\omega t} \right)$$

and

$$p(t) = \dot{q}(t) = -i \sqrt{\frac{\hbar\omega}{2}} \left(a e^{-i\omega t} - a^\dagger e^{i\omega t} \right) .$$

5. Quantization of the Scalar Field:

Continuum limit of the lattice:

$$\pi(\vec{x},t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\vec{x},t)} = \dot{\phi}(\vec{x},t) \; ,$$

$$H = \int d^3x \left[\pi\dot{\phi} - \mathcal{L}\right] \; ,$$

$$\left[\phi(\vec{x}\,',t)\,,\,\phi(\vec{x},t)\right] = 0 \quad \text{and} \quad \left[\pi(\vec{x}\,',t)\,,\,\pi(\vec{x},t)\right] = 0 \; ,$$

and

$$\left[\phi(\vec{x}\,',t)\,,\,\pi(\vec{x},t)\right] = i\hbar\,\delta(\vec{x}-\vec{x}\,') \; .$$

Use these relations to build the quantum theory.

Fourier transform:

$$\tilde{\phi}(\vec{k},t) \equiv \int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\; \phi(\vec{x},t) \; ,$$

and

$$\tilde{\pi}(\vec{k},t) \equiv \int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\; \pi(\vec{x},t) \; ,$$

Fourier inversion:

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\; \tilde{\phi}(\vec{k},t)$$

and

$$\pi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\; \tilde{\pi}(\vec{k},t) \; .$$

$\phi(\vec{x},t)$ real $\Longrightarrow$ Hermitian quantum operator $\Longrightarrow$

$$\tilde{\phi}(-\vec{k},t) = \phi^\dagger(\vec{k},t) \; ,$$

$$\tilde{\pi}(-\vec{k},t) = \pi^\dagger(\vec{k},t) \; .$$

Heisenberg equations of motion:

$$\frac{\partial^2\phi}{\partial t^2} - \vec{\nabla}^2\phi + m^2\phi = 0 \; .$$

Since $\hbar \neq 1$, m has the units of an inverse length, not a mass. Fourier transform obeys

$$\frac{\partial^2 \tilde{\phi}}{\partial t^2}(\vec{k}, t) + (\vec{k}^2 + m^2)\,\tilde{\phi}(\vec{k}, t) = 0 \ .$$

General solution:

$$\tilde{\phi}(\vec{k}, t) = \phi_1(\vec{k}) e^{-i\omega_p t} + \phi_2(\vec{k}) e^{i\omega_p t} \ ,$$

where

$$\omega_p = \sqrt{\vec{k}^2 + m^2} \ .$$

Reality $\tilde{\phi}(-\vec{k}, t) = \phi^\dagger(\vec{k}, t)$ implies

$$\phi_2(\vec{k}) = \phi_1^\dagger(-\vec{k}) \ ,$$

so

$$\tilde{\phi}(\vec{k}, t) = \phi_1(\vec{k}) e^{-i\omega_p t} + \phi_1^\dagger(-\vec{k}) e^{i\omega_p t} \ .$$

Since $\pi = \dot{\phi}$,

$$\tilde{\pi}(\vec{k}, t) = -i\omega_p \phi_1(\vec{k}) e^{-i\omega_p t} + i\omega_p \phi_1^\dagger(-\vec{k}) e^{i\omega_p t} \ .$$

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These two equations can be solved simultaneously to give:

$$\begin{aligned}\phi_1(\vec{k}) &= \frac{1}{2}\left[\tilde{\phi}(\vec{k}, t) + \frac{i}{\omega_p}\tilde{\pi}(\vec{k}, t)\right] e^{i\omega_p t} \\ &= \frac{1}{2}\int \mathrm{d}^3x\, e^{-i(\vec{k}\cdot\vec{x} - \omega_p t)} \left[\phi(\vec{x}, t) + \frac{i}{\omega_p}\,\pi(\vec{x}, t)\right] \ ,\end{aligned}$$

Using canonical commutation relations,

$$\begin{aligned}\left[\phi_1(\vec{k}),\, \phi_1(\vec{q})\right] &= \frac{1}{4}\int \mathrm{d}^3x \int \mathrm{d}^3y\, e^{-i(\vec{k}\cdot\vec{x} - \omega_k t)}\, e^{-i(\vec{q}\cdot\vec{y} - \omega_q t)} \\ &\qquad \times \left[\phi(\vec{x}, t) + \frac{i}{\omega_k}\,\pi(\vec{x}, t),\, \phi(\vec{y}, t) + \frac{i}{\omega_q}\,\pi(\vec{y}, t)\right] \\ &= \frac{1}{4}\int \mathrm{d}^3x \int \mathrm{d}^3y\, e^{-i(\vec{k}\cdot\vec{x} - \omega_k t)}\, e^{-i(\vec{q}\cdot\vec{y} - \omega_q t)} \\ &\qquad \times \left\{\frac{i}{\omega_q}\, i\hbar\, \delta^3(\vec{x} - \vec{y}) - \frac{i}{\omega_k}\, i\hbar\, \delta^3(\vec{x} - \vec{y})\right\} \\ &= \ldots \ .\end{aligned}$$

Using canonical commutation relations,

$$\left[\phi_1(\vec{k}\,),\,\phi_1(\vec{q}\,)\right] = \frac{1}{4}\int \mathrm{d}^3x \int \mathrm{d}^3y\, e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\, e^{-i(\vec{q}\cdot\vec{y}-\omega_q t)}$$

$$\times\left[\phi(\vec{x},t)+\frac{i}{\omega_k}\,\pi(\vec{x},t)\,,\,\phi(\vec{y},t)+\frac{i}{\omega_q}\,\pi(\vec{y},t)\right]$$

$$= \frac{1}{4}\int \mathrm{d}^3x \int \mathrm{d}^3y\, e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\, e^{-i(\vec{q}\cdot\vec{y}-\omega_q t)}$$

$$\times\left\{\frac{i}{\omega_q}\,i\hbar\,\delta^3(\vec{x}-\vec{y}\,) - \frac{i}{\omega_k}\,i\hbar\,\delta^3(\vec{x}-\vec{y}\,)\right\}$$

$$= \frac{\hbar}{4}\int \mathrm{d}^3x\, e^{-i\left((\vec{k}+\vec{q}\,)\cdot\vec{x}-(\omega_k+\omega_q)t\right)}\left\{\frac{1}{\omega_k}-\frac{1}{\omega_q}\right\}\,.$$

But

$$\int \mathrm{d}^3x\, e^{-i(\vec{k}+\vec{q}\,)\cdot\vec{x}} = (2\pi)^3\,\delta^3(\vec{k}+\vec{q}\,)\,,$$

which requires $\vec{q} = -\vec{k}$, but then the factor in curly brackets vanishes. Thus,

$$\boxed{\left[\phi_1(\vec{k}\,),\,\phi_1(\vec{q}\,)\right] = 0\,,}$$

as expected if $\phi_1(\vec{k}\,)$ is proportional to an annihilation operator.

Similarly,

$$\left[\phi_1(\vec{k})\,,\,\phi_1^\dagger(\vec{q})\right] = \frac{\hbar}{4}\int \mathrm{d}^3x\, e^{-i\left((\vec{k}-\vec{q})\cdot\vec{x}-(\omega_k-\omega_q)t\right)}\left\{\frac{1}{\omega_k}+\frac{1}{\omega_q}\right\}\;.$$

$$= \frac{\hbar}{4}\,(2\pi)^3\,\delta^3(\vec{k}-\vec{q})\,\frac{2}{\omega_k} = \frac{\hbar}{2\omega_k}\,(2\pi)^3\,\delta^3(\vec{k}-\vec{q})\;.$$

Continuum normalization convention:

Define

$$a(\vec{k}) = \sqrt{\frac{2\omega_k}{\hbar}}\,\phi_1(\vec{k})\;.$$

so that

$$\left[a(\vec{k})\,,\,a(\vec{q})\right] = 0\;,\quad \left[a(\vec{k})\,,\,a^\dagger(\vec{q})\right] = (2\pi)^3\,\delta^3(\vec{k}-\vec{q})\;,$$

Then

$$\phi(\vec{x},t) = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\sqrt{\frac{\hbar}{2\omega_k}}\left\{a(\vec{k})\,e^{i(\vec{k}\cdot x-\omega_k t)} + a^\dagger(\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\right\}\;.$$

Note: in 2nd term, I changed variables of integration $\vec{k}\longrightarrow-\vec{k}$,

Canonical momentum density:

$$\pi(\vec{x},t) = \dot{\phi}(\vec{x},t)$$

$$= -i\int\frac{\mathrm{d}^3k}{(2\pi)^3}\sqrt{\frac{\hbar\omega_k}{2}}\left\{a(\vec{k})\,e^{i(\vec{k}\cdot x-\omega_k t)} - a^\dagger(\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega_k t)}\right\}\;.$$

Creation and annihilation operators:

$$a(\vec{k}) = \sqrt{\frac{\omega_k}{2\hbar}} \int \mathrm{d}^3x\, e^{-i(\vec{k}\cdot\vec{x}-\omega_p t)} \left[\phi(\vec{x},t) + \frac{i}{\omega_p}\,\pi(\vec{x},t)\right] .$$

$$a^\dagger(\vec{k}) = \sqrt{\frac{\omega_k}{2\hbar}} \int \mathrm{d}^3x\, e^{i(\vec{k}\cdot\vec{x}-\omega_p t)} \left[\phi(\vec{x},t) - \frac{i}{\omega_p}\,\pi(\vec{x},t)\right] .$$

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