The Dirac Field

Part III: Dirac Equation, Lagrangian, Hamiltonian, Weyl Fields, Propagator

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WHAT HAVE WE LEARNED?

1) We learned that spin-$\frac{1}{2}$ particles cannot be bosons, but can be fermions.

2) From Eq. (100), $|\beta_L|^2 = |\beta_R|^2 = 1$, we learned that antiparticles are mandatory. In the equal-time commutator, the antiparticle contribution canceled the particle contribution, but only if the antiparticle creation/annihilation operators are included in the field with the same magnitude as the particle operators.

3) We found that the phases of β_L and β_R had to be equal to each other, but were undetermined. This freedom to rotate both phases together should have been expected: it corresponds to changing the phase of all antiparticle states. We never defined those phases in the first place, so it should make no difference if they are changed. (All antiparticle states must have their phases changed in the same way, however, or else the representation of the Poincaré group would have to be changed.)

Why is $P^0 > 0$ Important?

If we did not insist that $P^0 > 0$, then we could have replaced the $b_s^\dagger(\vec{p})$ in the expression for the field,

$$\psi_a(\vec{x},0) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left\{ a_s(\vec{p})\, u_a^s(\vec{p})\, e^{-ip_\mu x^\mu} + b_s^\dagger(\vec{p})\, v_a^s(\vec{p})\, e^{ip_\mu x^\mu} \right\} ,$$

by $b_s(\vec{p})$. That is, we could use an operator that, instead of creating an antiparticle with positive energy, would destroy a particle with negative energy. This interchange of b_s and $b_s^\dagger$ would reverse the sign of $\left[b_s^\dagger(-\vec{p})\,,\, b_s(-\vec{q})\right]$ that appeared in Eq. (93), allowing the bosonic commutator to vanish for spacelike separations. Such negative energy particles, however, apparently do not exist.

THE DIRAC EQUATION!

In this discussion, the Dirac equation is a byproduct. Now that we have constructed the Dirac field as a quantum operator, we can notice that it satisfies the Dirac equation.

Recall from Eqs. (76) and (77) that

$$u_L^s(\vec{q}=0) = u_R^s(\vec{q}=0) = \sqrt{m}\,\xi^s \ , \tag{103}$$

and that

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{104}$$

Thus γ^0 exchanges L and R, and for $\vec{q}=0$ the $u's$ are equal. Thus

$$\gamma^0\, u^s(\vec{q}=0) = u^s(\vec{q}=0) \ .$$

$$\gamma^0 \, u^s(\vec{q}=0) = u^s(\vec{q}=0) \; .$$

Since $q^\mu = (m, 0, 0, 0)$, this can be rewritten as

$$q_\mu \gamma^\mu \, u^s(\vec{q}=0) = m u^s(\vec{q}=0) \quad .$$

We can show that it holds in all frames by applying $\Lambda_{\frac{1}{2}}(B_{\vec{p}})$ to both sides, using

$$\Lambda_{\frac{1}{2}}(B_{\vec{p}}) u^s(\vec{q}=0) = u^s(\vec{p}) \tag{105}$$

and

$$\Lambda_{\frac{1}{2}} \gamma^\mu \Lambda_{\frac{1}{2}}^{-1} = \left[\Lambda^{-1}\right]^\mu{}_\nu \, \gamma^\nu \quad , \tag{106}$$

which follows from the commutation relations between γ^μ and the Lorentz generators, which show that γ^μ transforms as a Lorentz 4-vector. Remember that $B_{\vec{p}} \, q = p$.

Thus

$$\Lambda_{\frac{1}{2}}(B_{\vec{p}}) q_\mu \gamma^\mu \, u^s(\vec{q}=0) = m u^s(\vec{p}) \; ,$$

where the LHS can be rewritten as

$$\begin{aligned} LHS &= q_\mu \, \Lambda_{\frac{1}{2}}(B_{\vec{p}}) \, \gamma^\mu \, \Lambda_{\frac{1}{2}}^{-1}(B_{\vec{p}}) \, \Lambda_{\frac{1}{2}}(B_{\vec{p}}) \, u^s(\vec{q}=0) \\ &= q_\mu \, \left[B_{\vec{p}}^{-1}\right]^\mu{}_\nu \, \gamma^\nu \, u^s(\vec{p}) \\ &= q \cdot (B_{\vec{p}}^{-1} \, \gamma) \, u^s(\vec{p}) = p \cdot \gamma \, u^s(\vec{p}) \end{aligned}$$

where in the last line we used the fact that a dot product is Lorentz invariant, so we can apply $B_{\vec{p}}$ to each factor. Finally,

$$\boxed{(\gamma \cdot p) \, u^s(\vec{p}) = m \, u^s(\vec{p}) \; .} \tag{107}$$

For the v's, we had from Eq. (86),

$$v_L^s(\vec{q}=0) = -v_R^s(\vec{q}=0) \; . \tag{108}$$

When we wrote Eq. (86) we allowed for an arbitrary multiplicative factor between $v_L^s(\vec{q}=0)$ and $v_R^s(\vec{q}=0)$, in the form of the β's, but later we found that causality required $\beta_L = \beta_R$, so Eq. (108) is mandatory. The calculation for the v's is otherwise identical, leading to

$$(\gamma \cdot p)\, v^s(\vec{p}) = -m\, v^s(\vec{p}) \; . \tag{109}$$

Summarizing, we have

$$\psi_a(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left\{ a_s(\vec{p})\, u_a^s(\vec{p})\, e^{-ip_\mu x^\mu} + b_s^\dagger(\vec{p})\, v_a^s(\vec{p})\, e^{ip_\mu x^\mu} \right\} \; , \tag{72}$$

$$(\gamma \cdot p)\, u^s(\vec{p}) = m\, u^s(\vec{p}) \; , \tag{107}$$

and

$$(\gamma \cdot p)\, v^s(\vec{p}) = -m\, v^s(\vec{p}) \; . \tag{109}$$

The application of ∂_μ to $\psi(x)$ brings down a factor of $-ip_\mu$ for the first term and ip_μ for the second term, so the Dirac field satisfies

$$\boldsymbol{(i\gamma^\mu \partial_\mu - m)\, \psi(x) = 0 \quad .} \tag{110}$$

The Dirac Equation in 2×2 Blocks

It is sometimes useful to use our representation of the γ matrices, and the definitions $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$ to write the Dirac equation as:

$$\begin{pmatrix} -m & i\sigma \cdot \partial \\ i\bar{\sigma} \cdot \partial & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0 \; . \tag{111}$$

In this form we see that the spatial derivative mixes the upper and lower (L and R) components, as we commented earlier that we expected, on the grounds that $(0, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$ can only produce $(\frac{1}{2}, 0)$ or $(\frac{1}{2}, 1)$.

The Dirac Lagrangian

The Dirac equation is

$$(i\gamma^\mu \partial_\mu - m)\, \psi(x) = 0 \quad . \tag{110}$$

This is a 4-component equation, and the Lagrangian must be a real scalar. The natural guess is therefore to contract these indices with $\psi^\dagger(x)$, where

$$\psi(x) \equiv \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} , \qquad \psi^\dagger(x) = \begin{pmatrix} \psi_1^\dagger(x) & \psi_2^\dagger(x) & \psi_3^\dagger(x) & \psi_4^\dagger(x) \end{pmatrix} \; . \tag{112}$$

Then we can try

$$\mathscr{L}_{\text{maybe}} = \psi^\dagger (i\gamma^\mu \partial_\mu - m)\psi \; . \tag{113}$$

In principle we should vary the real and imaginary (e.g., hermitian and anti-hermitian) parts of $\psi(x)$, but we can equivalently vary ψ and $\psi^\dagger$ independently by defining

$$\bar{\partial}_{\psi_a} \equiv \frac{1}{2}\left(\frac{\partial}{\partial\,\mathrm{Re}\,\psi_a}\right) - i\left(\frac{\partial}{\partial\,\mathrm{Im}\,\psi_a}\right)$$
$$\bar{\partial}_{\psi_a^\dagger} \equiv \frac{1}{2}\left(\frac{\partial}{\partial\,\mathrm{Re}\,\psi_a}\right) + i\left(\frac{\partial}{\partial\,\mathrm{Im}\,\psi_a}\right) \qquad (114)$$

If we vary

$$\mathcal{L}_{\rm maybe} = \psi^\dagger(i\gamma^\mu\partial_\mu - m)\psi$$

with respect to $\psi^\dagger$, one gets the Dirac equation. Nonetheless, Peskin & Schroeder reject $\mathcal{L}_{\rm maybe}$ because it is not Lorentz-invariant.

This reason for rejecting $\mathcal{L}_{\rm maybe}$ seems weak to me, since it did generate the correct, Lorentz-invariant Dirac equation.

Rejecting $\mathcal{L}_{\rm maybe}$

A more serious problem with $\mathcal{L}_{\rm maybe}$ is that it gives inconsistent equations. Varying $\mathcal{L}_{\rm maybe}$ with respect to ψ, one integrates the action by parts and finds

$$\mathcal{L}_{\rm maybe} = \psi^\dagger(i\gamma^\mu\partial_\mu - m)\psi \quad \Longrightarrow$$
$$\psi^\dagger(-i\gamma^\mu\overleftarrow{\partial}_\mu - m) = 0\,. \qquad (115)$$

To compare with the usual Dirac equation, we have to take the complex conjugate (or adjoint) of this equation. To do this we need to compute $(\gamma^\mu)^\dagger$. For our conventions

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\,. \qquad (116)$$

One sees that

$$\left(\gamma^0\right)^\dagger = \gamma^0 \qquad \left(\gamma^i\right)^\dagger = -\gamma^i \ . \tag{117}$$

These signs are actually dictated by the original Dirac anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. These imply that $(\gamma^0)^2 = 1$, and $(\gamma^i)^2 = -1$, so they must be hermitian and antihermitian respectively. The anticommutation relations also imply that γ^μ and γ^ν anticommute when $\mu \neq \nu$. Thus we can write for all μ that:

$$\gamma^0 \left(\gamma^\mu\right)^\dagger \gamma^0 = \gamma^\mu \ . \tag{118}$$

So, taking the adjoint of Eq. (115),

$$0 = \left[\psi^\dagger(-i\gamma^\mu \overleftarrow{\partial}_\mu - m)\right]^\dagger = \left(i\left(\gamma^\mu\right)^\dagger \partial_\mu - m\right)\psi = (i\gamma^0\gamma^\mu\gamma^0 - m)\psi \ . \tag{119}$$

This is **not** the Dirac equation.

Note that this is a counterexample to the widely believed falsehood that equations derived from a Lagrangian are necessarily consistent.

Reality of $\mathcal{L}_{\text{maybe}}$

Root Problem: $\mathcal{L}_{\text{maybe}}$ is not ***real.***

To understand, think about minimizing a function of a complex vector $(z_1, \ldots, z_N)$:

$$L = z_a^* \, M^{ab} \, z_b \ ,$$

where M^{ab} is a matrix. We can vary with respect to z or z^* by defining

$$\bar{\partial}_{z_a} \equiv \frac{1}{2}\left(\frac{\partial}{\partial \operatorname{Re} z_a} - i\frac{\partial}{\partial \operatorname{Im} z_a}\right)$$

$$\bar{\partial}_{z_a^*} \equiv \frac{1}{2}\left(\frac{\partial}{\partial \operatorname{Re} z_a} + i\frac{\partial}{\partial \operatorname{Im} z_a}\right) \ .$$

Setting

$$\partial_{z_a^*} L = 0 \quad \Longrightarrow \quad M^{ab} z_b = 0 \ ,$$

and setting

$$\partial_{z_a} L = 0 \quad \Longrightarrow \quad z_a^* M^{ab} = 0 \ .$$

These two equations are consistent if and only if M^{ab} is hermitian, $M^{ab\,*} = M^{ba}$. This is also the condition that L be real.

Trying to make a complex L stationary is like trying to make two Lagrangians — i.e., the real and imaginary parts — stationary at the same time. It will usually be inconsistent.

Fixing the Dirac Lagrangian

To cure the problem, introduce a factor of γ^0 to cancel the γ^0's we found in Eq. (119):

$$\left[\psi^\dagger(-i\gamma^\mu \overleftarrow{\partial}_\mu - m)\right]^\dagger = (i\gamma^0\gamma^\mu\gamma^0 - m)\psi \ .$$

Let us try

$$\mathscr{L}_{\textbf{Dirac}} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi \ , \tag{120}$$

where

$$\bar{\psi}(x) = \psi^\dagger(x)\,\gamma^0 \qquad \left(\bar{\psi}_a(x) = \psi_b^\dagger(x)\,\gamma_{ba}^0\right) \ . \tag{121}$$

Then we can show that the action S is real:

$$S^\dagger = \int d^4x \, \left[\bar\psi \, (i\gamma^\mu \partial_\mu - m) \, \psi\right]^\dagger = \int d^4x \, \left[\psi^\dagger \, \gamma^0 \, (i\gamma^\mu \, \partial_\mu - m) \, \psi\right]^\dagger \ .$$

Integrating by parts,

$$\begin{aligned} S^\dagger &= \int d^4x \, \left[\psi^\dagger \, \gamma^0 \left(-i\gamma^\mu \overleftarrow{\partial}_\mu - m\right) \psi\right]^\dagger \\ &= \int d^4x \, \psi^\dagger \, \left(i\gamma^{\mu\dagger} \, \partial_\mu - m\right) \, \gamma^0 \, \psi \\ &= \int d^4x \, \psi^\dagger \, \gamma^0 \, \left(i\gamma^0 \, \gamma^{\mu\dagger} \, \gamma^0 \, \partial_\mu - m\right) \, \psi \\ &= \int d^4x \, \bar\psi \, (i\gamma^\mu \, \partial_\mu - m) \, \psi = S \ . \end{aligned} \tag{122}$$

Note that in the last line we used Eq. (118), $\gamma^0 \, (\gamma^\mu)^\dagger \, \gamma^0 = \gamma^\mu$.

Similarly we can show that the equations of motion are now consistent, since the equation obtained by varying ψ is

$$\bar\psi \left(-i\gamma^\mu \overleftarrow{\partial}_\mu - m\right) = 0 \ . \tag{122}$$

The adjoint of this equation is

$$\begin{aligned} 0 &= \left[\bar\psi \left(-i\gamma^\mu \overleftarrow{\partial}_\mu - m\right)\right]^\dagger \\ &= \left(i\gamma^{\mu\dagger} \partial_\mu - m\right) \gamma^{0\dagger} \psi \\ &= \gamma^0 \, \gamma^0 \, \left(i\gamma^{\mu\dagger} \partial_\mu - m\right) \gamma^0 \psi \\ &= \gamma^0 \, (i\gamma^\mu \partial_\mu - m) \, \psi \ . \end{aligned} \tag{123}$$

This is exactly the Dirac equation, multiplied by the (invertible) matrix γ^0.

Weyl Spinors

We already learned how to write the Dirac equation in 2×2 blocks:

$$\begin{pmatrix} -m & i\sigma \cdot \partial \\ i\bar{\sigma} \cdot \partial & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0 \ . \tag{111}$$

For the special case of $m = 0$, the two pieces decouple, giving the Weyl equations:

$$i\bar{\sigma} \cdot \partial \psi_L = 0 \,, \qquad i\sigma \cdot \partial \psi_R = 0 \ . \tag{124}$$

$\psi_L(x)$ and $\psi_R(x)$ are called Weyl fields. Since $\sigma^2 \, \bar{\sigma}^{\mu *} \, \sigma^2 = \sigma^\mu$, one can define

$$\chi_R = \sigma^2 \psi_R^\dagger \quad \Longrightarrow \quad i\bar{\sigma} \cdot \partial \chi_R = 0 \ . \tag{125}$$

The Weyl fields can be extracted from the Dirac field by defining

$$\begin{aligned} \gamma^5 &\equiv i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 \\ &= -\frac{i}{4!}\epsilon_{\mu\nu\lambda\sigma}\,\gamma^\mu\,\gamma^\nu\,\gamma^\lambda\,\gamma^\sigma \ , \end{aligned} \tag{126}$$

where $\epsilon_{\mu\nu\lambda\sigma}$ is the fully antisymmetric Levi-Civita tensor, with the sign convention (following P&S)

$$\epsilon_{0123} = -1 \ . \tag{127}$$

Note that this corresponds to $\epsilon^{0123} = 1$. γ^5 is Lorentz-invariant, which one can see by using the fact that $\epsilon_{\mu\nu\lambda\sigma}$ is Lorentz-invariant, or by noting that

$$\left[S^{\mu\nu} \,, \gamma^5 \right] = 0 \ . \tag{128}$$

where $S^{\mu\nu}$ are the generators of Lorentz transformations defined in Eq. (45).

In our conventions

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{129}$$

which clearly separates the upper and lower halves of the Dirac field:

$$\begin{aligned} \frac{1}{2}(1+\gamma^5)\psi &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} \\ \frac{1}{2}(1-\gamma^5)\psi &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} . \end{aligned} \tag{130}$$

For all choices of γ-matrices, γ^5 is defined by Eq. (126); it is Lorentz-invariant, and can be used to project L and R components of ψ (which are defined by the $(\frac{1}{2},0)+(0,\frac{1}{2})$ decomposition, but need not be the upper and lower pieces of ψ).

Behavior of Weyl (massless) spinors:

Recall that

$$u^s(\vec p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} , \qquad v^s(\vec p) = \begin{pmatrix} \sqrt{p\cdot\sigma}(-i\sigma^2\,\xi^s) \\ -\sqrt{p\cdot\bar\sigma}(-i\sigma^2\,\xi^s) \end{pmatrix} .$$

For $m = 0$ we can consider the case $p^\mu = (E,0,0,E)$, and we recall that $\sigma^\mu = (1,\sigma^i)$ and $\bar\sigma^\mu = (1,-\sigma^i)$. Then

$$p\cdot\sigma = p^0 - p^3\sigma^z = E(1-\sigma^z) = 2EP_- ,$$

where $P_- = \frac{1}{2}(1-\sigma^z)$ is the projector onto $\sigma^z = -1$ states, or equivalently the projector onto negative helicity states. Similarly, $p\cdot\bar\sigma = 2EP_+$, where P_+ projects onto $\sigma^z = 1$, or positive helicity states. Since σ^2 anticommutes with σ^3, one has $P_+\sigma^2 = \sigma^2 P_-$, and $P_-\sigma^2 = \sigma^2 P_+$. Putting all this together,

$$u^s(\vec p) = \sqrt{2E}\,\begin{pmatrix} P_-\,\xi^s \\ P_+\,\xi^s \end{pmatrix} , \qquad v^s(\vec p) = -i\sqrt{2E}\,\begin{pmatrix} \sigma^2\,P_+\,\xi^s \\ -\sigma^2\,P_-\,\xi^s \end{pmatrix} . \tag{131}$$

$$u^s(\vec{p}) = \sqrt{2E}\begin{pmatrix} P_-\,\xi^s \\ P_+\,\xi^s \end{pmatrix}\ , \qquad v^s(\vec{p}) = -i\sqrt{2E}\begin{pmatrix} \sigma^2\,P_+\,\xi^s \\ -\sigma^2\,P_-\,\xi^s \end{pmatrix}\ . \tag{131}$$

Recall that

$$\psi_a(x) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_s \left\{a_s(\vec{p})\,u^s_a(\vec{p})\,e^{-ip_\mu x^\mu} + b^\dagger_s(\vec{p})\,v^s_a(\vec{p})\,e^{ip_\mu x^\mu}\right\}\ ,$$

so one sees that for massless particles the upper components, ψ_L, destroy negative helicity particles (a_s destroys particles), and creates positive helicity antiparticles. For ψ_R it is the reverse.

For $m = 0$, one can have a theory with just negative helicity particles have and positive helicity antiparticles, or vice versa. One could have particles of both helicities, but there is no need for both to build a Lorentz-invariant field theory.

The Dirac Hamiltonian

The Dirac Lagrangian was given as Eq. (120),

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}\,(i\,\gamma^\mu\,\partial_\mu - m)\,\psi\ .$$

Following the canonical procedure,

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\psi^\dagger\ , \text{ and } H = \int d^3x\left(\pi\,\dot{\psi} - \mathcal{L}\right)\ . \tag{132}$$

Being careful with the signs,

$$i\,\gamma^\mu\,\partial_\mu = i\,\left(\gamma^0\,\partial_0 + \gamma^i\,\partial_i\right) = i\,\left(\gamma^0\,\partial_0 + \vec{\gamma}\cdot\vec{\nabla}\right)\ .$$

Then

$$H = \int d^3x\,\bar{\psi}\left(-i\vec{\gamma}\cdot\vec{\nabla} + m\right)\psi\ . \tag{133}$$

$$H = \int \mathrm{d}^3x\, \bar{\psi} \left(-i\vec{\gamma}\cdot\vec{\nabla} + m\right) \psi\ . \qquad (133)$$

Expanding the field, as in Eq. (92),

$$\psi_a(\vec{x},0) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left\{ a_s(\vec{p})\, u_a^s(\vec{p})\, e^{-ip_\mu x^\mu} + b_s^\dagger(\vec{p})\, v_a^s(\vec{p})\, e^{ip_\mu x^\mu} \right\}\Big|_{t=0}$$

$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left\{ a_s(\vec{p})\, u_a^s(\vec{p}) + b_s^\dagger(-\vec{p})\, v_a^s(-\vec{p}) \right\} e^{i\vec{p}\cdot\vec{x}}\ .$$

Inserting into Eq. (133),

$$H = \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \int \frac{\mathrm{d}^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{q}}}} \times$$

$$\times \sum_{rs} \left\{ a_s^\dagger(\vec{p})\, \bar{u}^s(\vec{p}) + b_s(-\vec{p})\, \bar{v}^s(-\vec{p}) \right\} e^{-i\vec{p}\cdot\vec{x}}$$

$$\times \left[-i\vec{\gamma}\cdot\vec{\nabla} + m\right] \left\{ a_r(\vec{q}) u^r(\vec{q}) + b_r^\dagger(-\vec{q}) v^r(-\vec{q}) \right\} e^{i\vec{q}\cdot\vec{x}}\ .$$

$$H = \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \int \frac{\mathrm{d}^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{q}}}} \times$$

$$\times \sum_{rs} \left\{ a_s^\dagger(\vec{p})\, \bar{u}^s(\vec{p}) + b_s(-\vec{p})\, \bar{v}^s(-\vec{p}) \right\} e^{-i\vec{p}\cdot\vec{x}}$$

$$\times \left[-i\vec{\gamma}\cdot\vec{\nabla} + m\right] \left\{ a_r(\vec{q}) u^r(\vec{q}) + b_r^\dagger(-\vec{q}) v^r(-\vec{q}) \right\} e^{i\vec{q}\cdot\vec{x}}\ .$$

One can replace $-i\vec{\gamma}\cdot\vec{\nabla}$ by $\vec{\gamma}\cdot\vec{q}$, and then integrate over $\vec{x}$, obtaining a $\delta(\vec{p}-\vec{q})$ which can be used to integrate over q:

$$H = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \sum_{rs} \left\{ a_s^\dagger(\vec{p})\, \bar{u}^s(\vec{p}) + b_s(-\vec{p})\, \bar{v}^s(-\vec{p}) \right\}$$

$$\times \left[\vec{\gamma}\cdot\vec{p} + m\right] \left\{ a_r(\vec{p}) u^r(\vec{p}) + b_r^\dagger(-\vec{p}) v^r(-\vec{p}) \right\}\ .$$

$$H = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \sum_{rs} \left\{ a_s^\dagger(\vec{p})\, \bar{u}^s(\vec{p}) + b_s(-\vec{p})\, \bar{v}^s(-\vec{p}) \right\}$$
$$\times \left[\vec{\gamma} \cdot \vec{p} + m \right] \left\{ a_r(\vec{p}) u^r(\vec{p}) + b_r^\dagger(-\vec{p}) v^r(-\vec{p}) \right\} \ .$$

Now use

$$(\gamma \cdot p)\, u^s(\vec{p}) = m\, u^s(\vec{p}) \ , \qquad (\gamma \cdot p)\, v^s(\vec{p}) = -m\, v^s(\vec{p}) \ ,$$

and

$$u_r^\dagger(\vec{p})\, u_s(\vec{p}) = 2E_{\vec{p}}\, \delta_{rs} \qquad v_r^\dagger(\vec{p})\, v_s(\vec{p}) = 2E_{\vec{p}}\, \delta_{rs}$$
$$u_r^\dagger(\vec{p})\, v_s(-\vec{p}) = 0 \qquad v_r^\dagger(\vec{p})\, u_s(-\vec{p}) = 0$$

to obtain

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \sum_s \left\{ a_s^\dagger(\vec{p})\, a_s(\vec{p}) - b_s(-\vec{p})\, b_s^\dagger(-\vec{p}) \right\} \ .$$

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \sum_s \left\{ a_s^\dagger(\vec{p})\, a_s(\vec{p}) - b_s(-\vec{p})\, b_s^\dagger(-\vec{p}) \right\} \ .$$

Now use $\vec{p} \rightarrow -\vec{p}$ in 2nd term, and reverse the order of the b and $b^\dagger$, using Eq. (97):

$$\left\{ b_s(\vec{p}) \, , \, b_r^\dagger(\vec{q}) \right\} = (2\pi)^3\, \delta_{rs} \delta^{(3)}(\vec{p} - \vec{q}) \ .$$

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \sum_s \left\{ a_s^\dagger(\vec{p})\, a_s(\vec{p}) + b_s^\dagger(\vec{p})\, b_s(\vec{p}) \right\} + E_{\text{vac}} \ , \tag{134}$$

where

$$E_{\text{vac}} = -2 \int d^3p\, E_{\vec{p}}\, \delta^{(3)}(\vec{0}) = -2 \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \times \text{Volume of space} \ , \tag{135}$$

where I am using

$$\delta^{(3)}(\vec{p}) = \int \frac{d^3x}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \ .$$

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \sum_s \left\{ a_s^\dagger(\vec{p})\, a_s(\vec{p}) + b_s^\dagger(\vec{p})\, b_s(\vec{p}) \right\} + E_{\text{vac}} \;, \tag{134}$$

where

$$E_{\text{vac}} = -2 \int d^3p\, E_{\vec{p}}\, \delta^{(3)}(\vec{0}) = -2 \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \times \text{Volume of space} \;. \tag{135}$$

Note that Fermi statistics caused the antiparticle energy to be positive (good!), and the vacuum energy to be negative (surprising?). The negative vacuum energy, although ill-defined, is still welcome: allows at least the hope that one might get the positive (bosonic) contributions to cancel against the negative (fermionic) contributions, giving an answer that is finite and hopefully small. Note that if we had 4 free scalar fields with the same mass, the cancelation would be exact: this is what happens in EXACTLY supersymmetric models, but it is spoiled as soon as the supersymmetry is broken.

Dirac Hole Theory

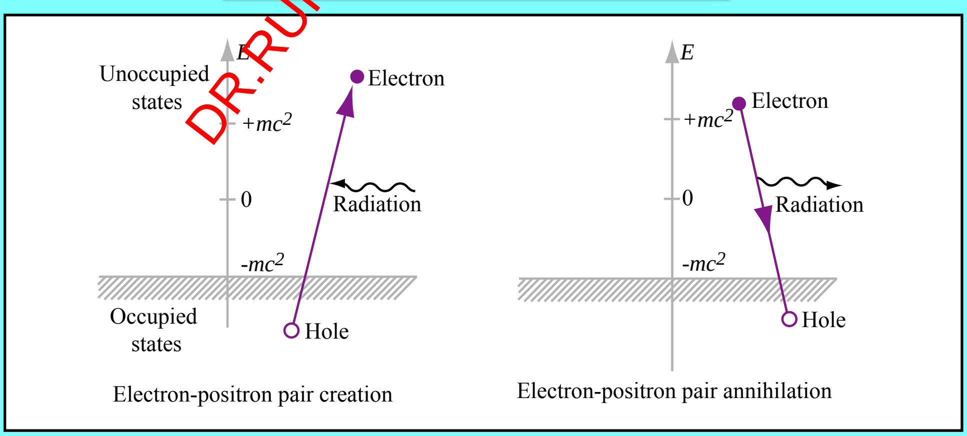

Figure by MIT OpenCourseWare. Adapted from Bjorken & Drell, vol. 1, p. 65.

In the 1-particle quantum mechanics formulation, positrons show up as negative energy states. Dirac proposed that in the vacuum, the negative energy "sea" was filled. Physical positrons, in this view, are holes in the Dirac sea. In QFT, on the other hand, particles and antiparticles are on equal footing. Nonetheless, the Dirac sea allows an intuitive way to understand the negative vacuum energy.

From Bjorken & Drell, vol. 1, p. 65

The Dirac Propagator

This is straightforward, so I will only summarize the results.

$$\begin{aligned}\langle 0\,|\psi_a(x)\,\bar\psi_b(y)|\,0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec p}}\sum_s u_a^s(\vec p)\,\bar u_b^s(\vec p)\,e^{-ip\cdot(x-y)}\\ &= (i\,\partial\!\!\!/_x + m)_{ab}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec p}}e^{-ip\cdot(x-y)}\\ &= (i\,\partial\!\!\!/_x + m)_{ab}\,D(x-y)\\ \langle 0\,|\bar\psi_b(x)\,\psi_a(y)|\,0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec p}}\sum_s v_a^s(\vec p)\,\bar v_b^s(\vec p)\,e^{-ip\cdot(y-x)}\\ &= -(i\,\partial\!\!\!/_x + m)_{ab}\,D(y-x)\ ,\end{aligned} \tag{136}$$

where $\partial\!\!\!/ = \gamma^\mu\,\partial_\mu$ and $D(x)$ is the scalar 2-point function $\langle 0\,|\phi(x)\phi(0)|\,0\rangle$.

The Retarded Dirac Propagator

$$\begin{aligned}S_R^{ab}(x-y) &\equiv \theta(x^0-y^0)\ \langle 0\,|\psi_a(x)\,\bar\psi_b(y)|\,0\rangle\\ &= (i\,\partial\!\!\!/_x + m)\,D_R(x-y)\ ,\end{aligned} \tag{137}$$

where $D_R(x-y)$ is the scalar retarded propagator. One can show

$$(i\,\partial\!\!\!/_x - m)S_R(x-y) = i\delta^{(4)}(x-y)\cdot 1_{4\times 4}\ . \tag{138}$$

The Fourier expansion is

$$S_R(x) = \int\frac{d^4p}{(2\pi)^4}\,e^{-ip\cdot x}\,\tilde S_R(p)\ ,\ \text{where}\ \ \tilde S_R(p) = \frac{i(\,p\!\!\!/ + m)}{(p^0+i\epsilon)^2 - |\vec p\,|^2 - m^2}\ . \tag{139}$$

The Feynman Propagator

$$S_F(x-y) \equiv \begin{cases} \langle 0 \left| \psi(x)\, \bar{\psi}(y) \right| 0 \rangle & \text{for } x^0 > y^0 \\ -\langle 0 \left| \bar{\psi}(y)\, \psi(x) \right| 0 \rangle & \text{for } y^0 > x^0 \end{cases}$$
$$\equiv \langle 0 \left| T\left\{ \psi(x)\, \bar{\psi}(y) \right\} \right| 0 \rangle \ . \qquad (140)$$

The Feynman propagator also satisfies Eq. (138). The Fourier expansion is

$$S_F(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x}\, \tilde{S}_F(p) \ , \ \text{where} \ \ \tilde{S}_F(p) = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \ . \qquad (141)$$