Discrete Symmetries of the Dirac Field

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Parity: $\vec{x} \longrightarrow -\vec{x}$

Group Properties:

Parity transformation matrix (on 4-vectors):

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (142)$$

Note that this is the transformation matrix, not the generator. Discrete transformations do not have generators.

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (142)$$

The Lorentz generators have the form

$$\vec{J} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{pmatrix} \qquad \vec{K} = \begin{pmatrix} 0 & x & x & x \\ x & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ x & 0 & 0 & 0 \end{pmatrix} \qquad (143)$$

where x denotes a possibly nonzero entry. This implies that

$$\left[\mathcal{P}\,,\,\vec{J}\right] = 0\;, \qquad \left\{\mathcal{P}\,,\,\vec{K}\right\} = 0\;. \qquad (144)$$

What about 4-momentum P^μ and parity $\mathcal{P}$?

Obviously,

$$\left\{\mathcal{P}\,,\,\vec{P}\right\} = 0\;, \qquad \left[\mathcal{P}\,,\,P^0\right] = 0\;. \qquad (145)$$

But what if we want to show this?

The complication is that we need a realization of the Poincaré group in which translations as well as boosts and rotations are represented linearly. Translations on coordinate 4-vectors are not linear. To find the commutators of P^μ, we used the representation of the operators on the Hilbert space of a scalar field.

The transformation properties are described by

$$\begin{aligned} \mathcal{P}^{-1}\,\phi(\vec{x},t)\,\mathcal{P} &= \phi(-\vec{x},t) \\ e^{-i\vec{P}_{\rm op}\cdot\vec{a}}\,\phi(\vec{x},t)\,e^{i\vec{P}_{\rm op}\cdot\vec{a}} &= \phi(\vec{x}+\vec{a},t) \\ e^{iP^0_{\rm op}t_0}\,\phi(\vec{x},t)\,e^{-iP^0_{\rm op}t_0} &= \phi(\vec{x},t+t_0)\ , \end{aligned} \tag{146}$$

where here $\mathcal{P}$ is shorthand for $U(\mathcal{P})$, the unitary operator in the Hilbert space that represents parity.

Note that P&S assume that $\mathcal{P}^2 = 1$, so $\mathcal{P}^{-1} = \mathcal{P}$, but I will not make this assumption. The argument that $\mathcal{P}^2 = 1$ is really the same as the argument that $e^{-2\pi i J_z} = 1$ (i.e., classically the transformation is the identity), and we know that $e^{-2\pi i J_z} \neq 1$. We will find that we can define phases so that $\mathcal{P}^2 = 1$, but it will **not** be possible to define phases so that $\mathcal{T}^2 = 1$, where $\mathcal{T}$ denotes time reversal.

To learn how $\mathcal{P}$ affects $\vec{P}$, define $\vec{P}^{\mathcal{P}}_{\rm op} \equiv \mathcal{P}^{-1}\,\vec{P}_{\rm op}\,\mathcal{P}$, Then

$$\begin{aligned} e^{-i\vec{P}^{\mathcal{P}}_{\rm op}\cdot\vec{a}}\,\phi(\vec{x},t)\,e^{i\vec{P}^{\mathcal{P}}_{\rm op}\cdot\vec{a}} &= \mathcal{P}^{-1}\,e^{-i\vec{P}_{\rm op}\cdot\vec{a}}\,\mathcal{P}\,\underbrace{\phi(\vec{x},t)}_{\mathcal{P}^{-1}\phi(-\vec{x},t)\,\mathcal{P}}\,\mathcal{P}^{-1}\,e^{i\vec{P}_{\rm op}\cdot\vec{a}}\,\mathcal{P} \\ &= \mathcal{P}^{-1}\,e^{-i\vec{P}_{\rm op}\cdot\vec{a}}\,\phi(-\vec{x},t)\,e^{i\vec{P}_{\rm op}\cdot\vec{a}}\,\mathcal{P} \\ &= \mathcal{P}^{-1}\,\phi(-\vec{x}+\vec{a},t)\,\mathcal{P} = \phi(\vec{x}-\vec{a},t)\ , \end{aligned} \tag{147}$$

so

$$\mathcal{P}^{-1}\,\vec{P}_{\rm op}\,\mathcal{P} = -\vec{P}_{\rm op}\quad . \tag{148}$$

Similarly one can show that

$$\mathcal{P}^{-1}\,P^0_{\rm op}\,\mathcal{P} = P^0_{\rm op}\quad . \tag{149}$$

Operation of $\mathcal{P}$ on States:

$\mathcal{P}$ anticommutes with $\vec{P}$ and $\vec{K}$, and commutes with $\vec{J}$. So

$$\mathcal{P}\,|\vec{p}{=}0, s{=}+\rangle = \eta_a\,|\vec{p}{=}0, s{=}+\rangle\ , \tag{150}$$

where η_a is an undetermined phase. In the rest frame we can use the standard nonrelativistic quantum theory description of rotations, so

$$|\vec{p}{=}0, s{=}-\rangle = (J_x - iJ_y)\,|\vec{p}{=}0, s{=}+\rangle\ ,$$

so

$$\mathcal{P}\,|\vec{p}{=}0, s{=}-\rangle = \eta_a\,|\vec{p}{=}0, s{=}-\rangle\ ,$$

with the same phase η_a. For an arbitrary momentum, $|\vec{q}, s\rangle = U(B_{\vec{q}})\,|\vec{p}{=}0, s\rangle$, so using the *canonical* representation of boosts

$$\begin{aligned}\mathcal{P}\,|\vec{q}, s\rangle &= \mathcal{P}\,U(B_{\vec{q}})\,\mathcal{P}^{-1}\,\mathcal{P}\,|\vec{p}{=}0, s\rangle = \eta_a\,U(B_{-\vec{q}})\,|\vec{p}{=}0, s\rangle\\ &= \eta_a\,|-\vec{q}, s\rangle\ .\end{aligned} \tag{151}$$

(It is more complicated in the helicity representation!)

For antiparticles the story repeats, but the phase is not necessarily the same:

$$\mathcal{P}\,|\vec{q}, s\rangle_b = \eta_b\,|-\vec{q}, s\rangle_b\ , \tag{152}$$

where the subscript b denotes an antiparticle state. Assuming further that multiparticle states transform as independent particles (i.e., all momenta are reversed, all spins remain the same, and each particle (antiparticle) contributes a factor η_a (η_b)), then

$$\begin{aligned}\mathcal{P}^{-1}\,a_s(\vec{p})\,\mathcal{P} &= \eta_a\,a_s(-\vec{p})\ , \quad \mathcal{P}^{-1}\,a_s^\dagger(\vec{p})\,\mathcal{P} = \eta_a^*\,a_s^\dagger(-\vec{p})\\ \mathcal{P}^{-1}\,b_s(\vec{p})\,\mathcal{P} &= \eta_b\,b_s(-\vec{p})\ , \quad \mathcal{P}^{-1}\,b_s^\dagger(\vec{p})\,\mathcal{P} = \eta_b^*\,b_s^\dagger(-\vec{p})\ .\end{aligned} \tag{153}$$

Operation of $\mathcal{P}$ on $\psi(x)$:

$$\psi_a(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left\{ a_s(\vec{p})\, u_a^s(\vec{p})\, e^{-ip_\mu x^\mu} + b_s^\dagger(\vec{p})\, v_a^s(\vec{p})\, e^{ip_\mu x^\mu} \right\} .$$

Then, using Eq. (153),

$$\mathcal{P}^{-1}\psi_a(\vec{x},t)\,\mathcal{P} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left\{ \eta_a\, a_s(-\vec{p})\, u_a^s(\vec{p})\, e^{-ip_\mu x^\mu} + \eta_b^*\, b_s^\dagger(-\vec{p})\, v_a^s(\vec{p})\, e^{ip_\mu x^\mu} \right\} .$$

Let $\tilde{x} \equiv (-\vec{x}, t)$, and change variables of integration by substituting $\vec{p}$ by $-\vec{p}$. Then

$$\mathcal{P}^{-1}\psi_a(\vec{x},t)\,\mathcal{P} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left\{ \eta_a\, a_s(\vec{p})\, u_a^s(-\vec{p})\, e^{-ip_\mu \tilde{x}^\mu} + \eta_b^*\, b_s^\dagger(\vec{p})\, v_a^s(-\vec{p})\, e^{ip_\mu \tilde{x}^\mu} \right\} .$$

Properties of $u^s(-\vec{p})$ and $v^s(-\vec{p})$:

From Eqs. (80) and (87),

$$u^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix}, \qquad v^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}(-i\sigma^2\,\xi^s) \\ -\sqrt{p\cdot\bar{\sigma}}(-i\sigma^2\,\xi^s) \end{pmatrix} , \qquad (154)$$

where we have set $\beta_L = \beta_R = 1$ Then

$$u^s(-\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\,\xi^s \\ \sqrt{p\cdot\sigma}\,\xi^s \end{pmatrix}, \qquad v^s(-\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}(-i\sigma^2\,\xi^s) \\ -\sqrt{p\cdot\sigma}(-i\sigma^2\,\xi^s) \end{pmatrix} .$$

Recalling that

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

one has

$$u^s(-\vec{p}) = \gamma^0\, u^s(\vec{p}), \qquad v^s(-\vec{p}) = -\gamma^0\, v^s(\vec{p}) , \qquad (155)$$

Back to Operation of $\mathcal{P}$ on $\psi(x)$:

$$\mathcal{P}^{-1}\psi_a(\vec{x},t)\,\mathcal{P} =$$
$$\gamma^0 \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_s \left\{\eta_a\, a_s(\vec{p})\, u_a^s(\vec{p})\, e^{-ip_\mu \tilde{x}^\mu} - \eta_b^*\, b_s^\dagger(\vec{p})\, v_a^s(\vec{p})\, e^{ip_\mu \tilde{x}^\mu}\right\} \,. \tag{156}$$

Requirement of Causality: ***IF*** the theory is going to be parity invariant, then

$$\mathcal{P}^{-1}\,\mathcal{L}\,\mathcal{P} = \mathcal{L}\,. \tag{157}$$

So if $\mathcal{L}$ contains ψ, it must also contain $\mathcal{P}^{-1}\,\psi\,\mathcal{P}$. But all terms in $\mathcal{L}$ must be mutually causal, meaning that they commute at spacelike separations. Thus $\mathcal{P}^{-1}\,\psi\,\mathcal{P}$ must anticommute with ψ and $\bar{\psi}$ at spacelike separations.

When we constructed ψ and insisted that $\bar{\psi}$ and ψ be causal, at Eq. (99), we found that the antiparticle contribution was constrained: $\beta_R = \beta_L$, with $|\beta_L| = |\beta_R| = 1$. But there was an arbitrary phase, which we fixed by choosing $\beta_L = \beta_R = 1$. But if we now calculate $\left\{\mathcal{P}^{-1}\psi_a(\vec{x},0)\,\mathcal{P}\,,\,\bar{\psi}(\vec{y},0)\right\}$, we will find that it will vanish only if $\eta_a = -\eta_b^*$. Thus, if parity is a valid symmetry, we require

$$\eta_a\,\eta_b = -1\,, \tag{158}$$

and then

$$\mathcal{P}^{-1}\psi(\vec{x},t)\,\mathcal{P} = \eta_a\,\gamma^0\,\psi(-\vec{x},t)\,. \tag{159}$$

Electrons and positrons have opposite intrinsic parity.

Parity and Dirac Field Bilinears

Since $\psi_a(x)$ and $\bar{\psi}_a(x)$ each have 4 components, there are 16 possible Dirac field bilinears. They can be arranged into 5 quantities with well-defined Lorentz transformation properties:

$$\bar{\psi}\psi, \quad \bar{\psi}\gamma^\mu\psi, \quad i\bar{\psi}\,[\gamma^\mu\,,\,\gamma^\nu]\,\psi, \quad \bar{\psi}\gamma^\mu\gamma^5\psi, \quad i\bar{\psi}\gamma^5\psi\;. \tag{160}$$

Each quantity shown is hermitian. The number of independent components is 1, 4, 6, 4, and 1, respectively, totaling 16 as expected.

Given that $\mathcal{P}^{-1}\psi(\vec{x},t)\mathcal{P} = \eta_a\,\gamma^0\,\psi(-\vec{x},t)$, it is straightforward to find that

$$\begin{aligned}
&\mathcal{P}^{-1}\bar{\psi}\psi\,\mathcal{P} = +\mathcal{O}(-\vec{x},t) && \mathcal{P}^{-1}i\bar{\psi}\gamma^5\psi\,\mathcal{P} = -\mathcal{O}(-\vec{x},t)\\
&\mathcal{P}^{-1}\bar{\psi}\gamma^i\psi\,\mathcal{P} = -\mathcal{O}(-\vec{x},t) && \mathcal{P}^{-1}\bar{\psi}\gamma^i\gamma^5\psi\,\mathcal{P} = +\mathcal{O}(-\vec{x},t)\\
&\mathcal{P}^{-1}\bar{\psi}\gamma^0\psi\,\mathcal{P} = +\mathcal{O}(-\vec{x},t) && \mathcal{P}^{-1}\bar{\psi}\gamma^0\gamma^5\psi\,\mathcal{P} = -\mathcal{O}(-\vec{x},t)\\
&\mathcal{P}^{-1}i\bar{\psi}\left[\gamma^0\,,\,\gamma^i\right]\psi\,\mathcal{P} = -\mathcal{O}(-\vec{x},t) && \mathcal{P}^{-1}i\bar{\psi}\left[\gamma^i\,,\,\gamma^j\right]\psi\,\mathcal{P} = +\mathcal{O}(-\vec{x},t)
\end{aligned} \tag{161}$$

where the quantity $\mathcal{O}$ on the right-hand side of each equation denotes the quantity between $\mathcal{P}^{-1}$ and $\mathcal{P}$ on the left.

Time Reversal: $t \longrightarrow -t$

Schrödinger Picture:

$$|\psi(t)\rangle = e^{-iHt}\,|\psi(0)\rangle \ . \tag{162}$$

We seek a time-reversed state

$$\left|\tilde\psi(t)\right\rangle = \mathcal{T}\left|\psi(-t)\right\rangle \ , \tag{163}$$

but we also insist that $\left|\tilde\psi(t)\right\rangle$ obey the Schrödinger equation:

$$\left|\tilde\psi(t)\right\rangle = e^{-iHt}\left|\tilde\psi(0)\right\rangle \ . \tag{164}$$

$$\left|\psi(t)\right\rangle = e^{-iHt}\left|\psi(0)\right\rangle , \left|\tilde\psi(t)\right\rangle = \mathcal{T}\left|\psi(-t)\right\rangle , \left|\tilde\psi(t)\right\rangle = e^{-iHt}\left|\tilde\psi(0)\right\rangle \ .$$

Furthermore, we insist in a time-reversal invariant theory that the same operator $\mathcal{T}$ can reverse the evolution of any state. Then

$$\mathcal{T}\left|\psi(-t)\right\rangle = \mathcal{T}\,e^{iHt}\left|\psi(0)\right\rangle \ . \tag{165}$$

But we also know that

$$\begin{aligned}\mathcal{T}\left|\psi(-t)\right\rangle &= \left|\tilde\psi(t)\right\rangle = e^{-iHt}\left|\tilde\psi(0)\right\rangle \\ &= e^{-iHt}\,\mathcal{T}\left|\psi(0)\right\rangle \ .\end{aligned} \tag{166}$$

Comparing the RHS's of (165) and (166),

$$\mathcal{T}\,e^{iHt} = e^{-iHt}\,\mathcal{T} \ . \tag{167}$$

Expanding to first order:

$$\mathcal{T} e^{iHt} = e^{-iHt}\mathcal{T} \quad \Longrightarrow \quad \mathcal{T}(iH) = (-iH)\mathcal{T} \ . \tag{168}$$

We know from Wigner's theorem that any symmetry operator (any mapping of rays in a Hilbert space onto rays in the same Hilbert space which preserves overlap probabilities) can always be represented as one of the following:

1) a linear and unitary operator:

$$\begin{aligned} U\left(\alpha\left|\psi_1\right\rangle + \beta\left|\psi_2\right\rangle\right) &= \alpha U\left|\psi_1\right\rangle + \beta U\left|\psi_2\right\rangle \\ \left\langle U\psi_2 \,\middle|\, U\psi_1\right\rangle &= \left\langle \psi_2 \,\middle|\, \psi_1\right\rangle \end{aligned} \tag{169}$$

2) an antilinear and antiunitary operator:

$$\begin{aligned} A\left(\alpha\left|\psi_1\right\rangle + \beta\left|\psi_2\right\rangle\right) &= \alpha^* A\left|\psi_1\right\rangle + \beta^* A\left|\psi_2\right\rangle \\ \left\langle A\psi_2 \,\middle|\, A\psi_1\right\rangle &= \left\langle \psi_2 \,\middle|\, \psi_1\right\rangle^* \ . \end{aligned} \tag{170}$$

Unitary or Antiunitary, That is the Question:

If $\mathcal{T}$ is unitary, $\mathcal{T}(iH) = (-iH)\mathcal{T}$ implies that $\mathcal{T}$ must anticommute with H. But then $\mathcal{T}$ would map each positive energy state into a corresponding negative energy state, violating the positivity of the energy. If we don't want a theory in which the eigenvalues of H are symmetric about zero, we must choose $\mathcal{T}$ to be antiunitary.

P&S say: "The way out is to retain the unitarity condition $\mathcal{T}^\dagger = \mathcal{T}$, but have $\mathcal{T}$ act on c-numbers as well as operators."

BUT: Antiunitary operators do NOT have adjoints, and $\mathcal{T}$ operates on the Hilbert space, not on c-numbers or operators. An adjoint is defined by

$$\left\langle \mathcal{O}^\dagger \psi_2 \,\middle|\, \psi_1 \right\rangle = \left\langle \psi_2 \,\middle|\, \mathcal{O}\psi_1 \right\rangle \ .$$

If $\mathcal{O}$ were antiunitary, the left-hand side would be linear in $|\psi_1\rangle$ and the right-hand side would be antilinear, so they could not possibly be equal. Eq. (170) correctly defines the properties of antiunitary operators:

$$\begin{aligned} A\left(\alpha\left|\psi_1\right\rangle + \beta\left|\psi_2\right\rangle\right) &= \alpha^* A\left|\psi_1\right\rangle + \beta^* A\left|\psi_2\right\rangle \\ \left\langle A\psi_2 \,\middle|\, A\psi_1\right\rangle &= \left\langle \psi_2 \,\middle|\, \psi_1\right\rangle^* \ . \end{aligned} \tag{170}$$

$\mathcal{T}$ and the Poincaré Algebra

The time reversal transformation on 4-vectors is given by

$$\mathcal{T} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{171}$$

The matrix defines a linear transformation, however, so it is not really the operator that we want to describe.

The properties of $\mathcal{T}$, now used exclusively to refer to the operator in the Hilbert space, are determined by

$$\mathcal{T}^{-1}\,\phi(\vec{x},t)\,\mathcal{T} = \phi(\vec{x},-t)\ , \tag{172}$$

where we note that antiunitary operators are invertible, but we should not assume that $\mathcal{T}^{-1} = \mathcal{T}$.

Under rotations

$$U^{-1}(R)\,\phi(\vec{x},t)\,U(R) = \phi(R^{-1}\vec{x},t)\ ,$$

which can be combined with Eq. (172) to show that $\mathcal{T}$ commutes with

$$U\big(R(\hat{n},\theta)\big) = e^{-i\theta\hat{n}\cdot\vec{J}}\ .$$

But since the exponent contains an i, $\mathcal{T}$ must anticommute with $\vec{J}$. The same is true for $\vec{P}$, the generator of translations.

For boosts,

$$U^{-1}(B)\,\phi(x)\,U(B) = \phi(B^{-1}x)\ ,$$

where

$$U\big(B(\hat{n},\xi)\big) = e^{-i\xi\hat{n}\cdot\vec{K}}\ .$$

To combine this with Eq. (172), we need to know what happens to $B^{-1}x$ when t is reversed. Defining T to be the matrix that reverses the time component of 4-vectors, one can show the 4-vector identity

$$B(\hat{n},\xi)\,T\,x = T\,B(-\hat{n},\xi)\,x\ ,$$

which can be used with the above equations to show that $\mathcal{T}$ reverses the direction of boosts, and hence $\mathcal{T}$ commutes with $\vec{K}$ (with the reversal attributed to the i in $e^{-i\xi\hat{n}\cdot\vec{K}}$).

Finally, then

$$\left\{\mathcal{T}\,,\,\vec{J}\right\} = 0 \qquad \left[\mathcal{T}\,,\,\vec{K}\right] = 0$$
$$\left\{\mathcal{T}\,,\,\vec{P}\right\} = 0 \qquad \left[\mathcal{T}\,,\,P^0\right] = 0\ . \qquad (173)$$

$\mathcal{T}$ Operating on Spin-$\frac{1}{2}$ States

Since $\mathcal{T}$ anticommutes with both J_z and $\vec{P}$, it must reverse both the spin and the momentum of the states on which it acts. So

$$\mathcal{T}\,|\vec{p}{=}0, s{=}+\rangle = \eta_t\,|\vec{p}{=}0, s{=}-\rangle \qquad (174)$$

For some phase η_t, where $|\eta_t| = 1$. Then

$$\begin{aligned}\mathcal{T}\,|\vec{p}{=}0, s{=}-\rangle &= \mathcal{T}\,(J_x - iJ_y)\,|\vec{p}{=}0, s{=}+\rangle\\ &= -(J_x + iJ_y)\,\mathcal{T}\,|\vec{p}{=}0, s{=}+\rangle\\ &= -\eta_t\,(J_x + iJ_y)\,|\vec{p}{=}0, s{=}-\rangle\\ &= -\eta_t\,|\vec{p}{=}0, s{=}+\rangle\end{aligned} \qquad (175)$$

Thus

$$\mathcal{T}^2 \left|\vec{p}{=}0, s{=}+\right\rangle = \mathcal{T}\eta_t \left|\vec{p}{=}0, s{=}-\right\rangle = \eta_t^* \mathcal{T} \left|\vec{p}{=}0, s{=}-\right\rangle$$
$$= -|\eta_t|^2 \left|\vec{p}{=}0, s{=}+\right\rangle = -\left|\vec{p}{=}0, s{=}+\right\rangle \ . \qquad (176)$$

Since $\mathcal{T}^2$ commutes with all the generators, this result holds for boosted states as well. $\mathcal{T}^2 = -1$ on spin-$\frac{1}{2}$ states.

On a general state, $\left|\vec{q}, s\right\rangle = U(B_{\vec{q}}) \left|\vec{p}{=}0, s\right\rangle$, in the canonical representation,

$$\mathcal{T}\left|\vec{q}, s{=}+\right\rangle = \mathcal{T}U(B_{\vec{q}})\left|\vec{p}{=}0, s{=}+\right\rangle$$
$$= U(B_{-\vec{q}})\,\mathcal{T}\left|\vec{p}{=}0, s{=}+\right\rangle$$
$$= \eta_t U(B_{-\vec{q}})\left|\vec{p}{=}0, s{=}-\right\rangle$$
$$= \eta_t \left|-\vec{q}, s{=}-\right\rangle \ . \qquad (177)$$

Similarly,

$$\mathcal{T}\left|\vec{q}, s{=}-\right\rangle = -\eta_t \left|-\vec{q}, s{=}+\right\rangle \ . \qquad (178)$$

$\mathcal{T}$ Operating on States (Cont.)

Summary of where we are:

$$\begin{aligned} \mathcal{T}\,|\vec{q}, s{=}+\rangle &= \eta_t\,|-\vec{q}, s{=}-\rangle \\ \mathcal{T}\,|\vec{q}, s{=}-\rangle &= -\eta_t\,|-\vec{q}, s{=}+\rangle \ . \end{aligned} \tag{179}$$

These two equations can be combined by writing

$$\mathcal{T}\,|\vec{q}, s\rangle = i\eta_t\sigma^2_{ss'}\,|-\vec{q}, s'\rangle \ , \ \text{where } i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \tag{180}$$

where we adopt the convention that repeated indices (such as s') are summed.

$\mathcal{T}$ and Creation/Annihilation Operators

Creation Operators Conjugated by $\mathcal{T}$: $\mathcal{T}^{-1}a^\dagger_s(\vec{p})\,\mathcal{T}$.

We know that $\mathcal{T}^2$ operating on single particle states is -1, so for single particle states $\mathcal{T}^{-1} = -\mathcal{T}$. The vacuum must transform under $\mathcal{T}$ into the vacuum (the energy cannot change), and we can choose the phase to be 1. Then

$$\begin{aligned} \mathcal{T}^{-1}a^\dagger_s(\vec{p})\,\mathcal{T}\,|0\rangle &= \mathcal{T}^{-1}a^\dagger_s(\vec{p})\,|0\rangle = \frac{1}{\sqrt{2E_{\vec{p}}}}\,\mathcal{T}^{-1}\,|\vec{p}, s\rangle \\ &= \frac{-i}{\sqrt{2E_{\vec{p}}}}\eta_t\sigma^2_{ss'}\,|-\vec{p}, s'\rangle = -i\eta_t\sigma^2_{ss'}a^\dagger_{s'}(-\vec{p})\,|0\rangle \ . \end{aligned} \tag{181}$$

$$\mathcal{T}^{-1} a_s^\dagger(\vec{p})\,\mathcal{T}\,|0\rangle = -i\eta_t \sigma^2_{ss'} a^\dagger_{s'}(-\vec{p})\,|0\rangle \ . \tag{181}$$

If we assume that the particles in multiparticle states transform independently,

$$\mathcal{T}\,|\vec{p}_1 s_1, \ldots, \vec{p}_N s_N\rangle = (i\eta_t)^N \sigma^2_{s_1 s_1'} \ldots \sigma^2_{s_N s_N'}\, |-\vec{p}\, s_1' \ldots, -\vec{p}_N s_N'\rangle \ , \tag{182}$$

then Eq. (181) can be promoted into an operator identity,

$$\mathcal{T}^{-1} a_s^\dagger(\vec{p})\,\mathcal{T} = -i\eta_t \sigma^2_{ss'} a^\dagger_{s'}(-\vec{p}) \ , \tag{183}$$

by the same methods that were used for Lorentz transformations in Eq. (59) (on 4/24/08).

Annihilation Operators Conjugated by $\mathcal{T}$: $\mathcal{T}^{-1} a_s(\vec{p})\,\mathcal{T}$.

To find the action of $\mathcal{T}$ on annihilation operators, we want to calculate the adjoint of Eq. (183). But we should be careful when we take the adjoint of an expression involving antiunitary operators. In general, the adjoint of an operator $\mathcal{O}$ is defined by the relation

$$\langle \psi_2 \,|\mathcal{O}\psi_1 \rangle = \langle \mathcal{O}^\dagger \psi_2 \,|\psi_1 \rangle \ . \tag{184}$$

Now suppose that L is an arbitrary linear operator and A is an arbitrary antilinear operator. Then

$$\begin{aligned} \langle \psi_2 \,|A^{-1} L A \psi \rangle &= \langle A\psi_2 \,|LA\psi \rangle^* \\ &= \langle L^\dagger A \psi_2 \,|A\psi \rangle^* \\ &= \langle A^{-1} L^\dagger A \psi_2 \,|\psi \rangle \ , \end{aligned} \tag{185}$$

Therefore

$$\left(A^{-1}LA\right)^{\dagger} = A^{-1}L^{\dagger}A\ , \tag{186}$$

and then Eq. (183) implies that

$$\mathcal{T}^{-1} a_s(\vec{p})\, \mathcal{T} = -i\eta_t^* \sigma^2_{ss'} a_{s'}(-\vec{p})\ . \tag{187}$$

(Note that $i\sigma^2$ is real, so it is unchanged by taking the adjoint.)

Time Reversal $\mathcal{T}$ and Antiparticles:

All the same arguments apply to antiparticles, so one has the same equations, except that the arbitrary phase need not be the same. So we introduce $\bar{\eta}_t$ ($|\bar{\eta}_t| = 1$) for antiparticles:

$$\mathcal{T}^{-1} b_s^{\dagger}(\vec{p})\, \mathcal{T} = -i\bar{\eta}_t \sigma^2_{ss'} b^{\dagger}_{s'}(-\vec{p})\ , \quad \mathcal{T}^{-1} b_s(\vec{p})\, \mathcal{T} = -i\bar{\eta}^*_t \sigma^2_{ss'} b_{s'}(-\vec{p})\ . \tag{188}$$

$\mathcal{T}$ and the Dirac Field

$$\psi_a(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left\{ a_s(\vec{p})\, u^s_a(\vec{p})\, e^{-ip_\mu x^\mu} + b^{\dagger}_s(\vec{p})\, v^s_a(\vec{p})\, e^{ip_\mu x^\mu} \right\}\ .$$

Using Eqs. (187) and (188),

$$\mathcal{T}^{-1}\psi_a(\vec{x},t)\,\mathcal{T} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \times \sum_s \left\{ -i\eta^*_t \sigma^2_{ss'} a_{s'}(-\vec{p})\, u^{s*}_a(\vec{p})\, e^{ip_\mu x^\mu} - i\bar{\eta}_t \sigma^2_{ss'} b^{\dagger}_{s'}(-\vec{p})\, v^{s*}_a(\vec{p})\, e^{-ip_\mu x^\mu} \right\}\ . \tag{189}$$

Now define $\tilde{x}_t \equiv (\vec{x}, -t)$, and replace the variable of integration $\vec{p}$ with $-\vec{p}$.

Now define $\tilde{x}_t \equiv (\vec{x}, -t)$, and replace the variable of integration $\vec{p}$ with $-\vec{p}$. Then

$$\mathcal{T}^{-1} \psi_a(\vec{x}, t)\, \mathcal{T} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \times$$
$$\sum_s \left\{ -i\eta_t^* \sigma^2_{ss'} a_{s'}(\vec{p})\, u_a^{s*}(-\vec{p})\, e^{-ip_\mu \tilde{x}_t^\mu} - i\bar{\eta}_t \sigma^2_{ss'} b^\dagger_{s'}(\vec{p})\, v_a^{s*}(-\vec{p})\, e^{ip_\mu \tilde{x}_t^\mu} \right\} . \tag{190}$$

To relate this expression to the original field $\psi_a(x)$, we need identities for the u and v functions (Eqs. (80) and (87)):

$$u^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} , \qquad v^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}(-i\sigma^2\,\xi^s) \\ -\sqrt{p\cdot\bar{\sigma}}(-i\sigma^2\,\xi^s) \end{pmatrix} , \tag{191}$$

where

$$\xi^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \xi^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{192}$$

Eq. (192) is equivalent to $\xi^s_a = \delta^s_a$, so we can write

$$u^s(\vec{p}) = \begin{pmatrix} u^s_{La}(\vec{p}) \\ u^s_{Ra}(\vec{p}) \end{pmatrix} = \begin{pmatrix} [\sqrt{p\cdot\sigma}\,]_{as} \\ [\sqrt{p\cdot\bar{\sigma}}\,]_{as} \end{pmatrix} , \tag{193}$$

and then

$$-iu^{s*}(-\vec{p})\sigma^2_{ss'} = -i \begin{pmatrix} u^{s*}_{La}(-\vec{p}) \\ u^{s*}_{Ra}(-\vec{p}) \end{pmatrix} \sigma^2_{ss'} = -i \begin{pmatrix} [(\sqrt{p\cdot\bar{\sigma}})^*\,\sigma^2]_{as'} \\ [(\sqrt{p\cdot\sigma})^*\,\sigma^2]_{as'} \end{pmatrix} \tag{194}$$

To continue the manipulations, we recall that we have for compactness defined $\sqrt{M}$, where M is a positive semidefinite matrix, by diagonalizing M and taking the square root of the eigenvalues. This quantity is hard to manipulate, since one can easily show by example that $\sqrt{M^*}$ is not necessarily equal to $\left(\sqrt{M}\right)^*$. So, to continue in a straightforward way, we rewrite $\sqrt{p\cdot\sigma}$ and $\sqrt{p\cdot\bar{\sigma}}$ in terms of the projection operators onto the $\hat{p}\cdot\vec{\sigma} = \pm 1$ eigenspaces of $\hat{p}\cdot\vec{\sigma}$:

$$\sqrt{p\cdot\sigma} = \left[\frac{1+\hat p\cdot\vec\sigma}{2}\right]\sqrt{p^0-|\vec p|} + \left[\frac{1-\hat p\cdot\vec\sigma}{2}\right]\sqrt{p^0+|\vec p|}$$
$$\sqrt{p\cdot\bar\sigma} = \left[\frac{1+\hat p\cdot\vec\sigma}{2}\right]\sqrt{p^0+|\vec p|} + \left[\frac{1-\hat p\cdot\vec\sigma}{2}\right]\sqrt{p^0-|\vec p|}\ . \qquad (195)$$

If you look back to the derivation of Eq. (80), 5/1/08, you will see that $\sqrt{p\cdot\sigma}$ and $\sqrt{p\cdot\bar\sigma}$ were introduced as abbreviations for the expressions on the RHS of Eq. (195). Thus, using the identity $\sigma^{i*} = -\sigma^2\sigma^i\sigma^2$, one has

$$\begin{aligned}(\sqrt{p\cdot\sigma})^* &= \sigma^2\left\{\left[\frac{1-\hat p\cdot\vec\sigma}{2}\right]\sqrt{p^0-|\vec p|} + \left[\frac{1+\hat p\cdot\vec\sigma}{2}\right]\sqrt{p^0+|\vec p|}\right\}\sigma^2\\ &= \sigma^2\sqrt{p\cdot\bar\sigma}\,\sigma^2\ ,\end{aligned} \qquad (196)$$

and one can similarly show that

$$(\sqrt{p\cdot\bar\sigma})^* = \sigma^2\sqrt{p\cdot\sigma}\,\sigma^2\ . \qquad (197)$$

–32–

These relations — $(\sqrt{p\cdot\sigma})^* = \sigma^2\sqrt{p\cdot\bar\sigma}\sigma^2$ and $(\sqrt{p\cdot\bar\sigma})^* = \sigma^2\sqrt{p\cdot\sigma}\sigma^2$ — can be used to simplify Eq. (194):

$$\begin{aligned}-iu^{s*}(-\vec p)\sigma^2_{ss'} &= -i\begin{pmatrix}[(\sqrt{p\cdot\bar\sigma})^*\,\sigma^2]_{as'}\\ [(\sqrt{p\cdot\sigma})^*\,\sigma^2]_{as'}\end{pmatrix} = -i\begin{pmatrix}[\sigma^2(\sqrt{p\cdot\sigma})\sigma^2\,\sigma^2]_{as'}\\ [\sigma^2(\sqrt{p\cdot\bar\sigma})\sigma^2\,\sigma^2]_{as'}\end{pmatrix}\\ &= -i\begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix}u(\vec p)\ .\end{aligned} \qquad (198)$$

In our conventions,

$$\gamma^\mu = \begin{pmatrix}0 & \sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix} \quad\Longrightarrow\quad \begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix} = -i\gamma^1\gamma^3\ , \qquad (199)$$

so

$$-iu^{s*}(-\vec p)\sigma^2_{ss'} = -\gamma^1\gamma^3 u^{s'}(\vec p)\ . \qquad (200)$$

For the v-function, one has the same result:

$$v^s(\vec p) = \begin{pmatrix} v^s_{La}(\vec p) \\ v^s_{Ra}(\vec p) \end{pmatrix} = \begin{pmatrix} [\sqrt{p\cdot\sigma}\,(-i\sigma^2)]_{as} \\ -[\sqrt{p\cdot\bar\sigma}\,(-i\sigma^2)]_{as} \end{pmatrix} \;, \tag{201}$$

and then

$$\begin{aligned} -iv^{s*}(-\vec p)\sigma^2_{ss'} &= -i\begin{pmatrix} [(\sqrt{p\cdot\bar\sigma})^*\,(-i\sigma^2)\sigma^2]_{as'} \\ -[(\sqrt{p\cdot\sigma})^*\,(-i\sigma^2)\sigma^2]_{as'} \end{pmatrix} \\ &= -i\begin{pmatrix} [\sigma^2\sqrt{p\cdot\sigma}\sigma^2\,(-i\sigma^2)\sigma^2]_{as'} \\ -[\sigma^2\sqrt{p\cdot\bar\sigma}\sigma^2\,(-i\sigma^2)\sigma^2]_{as'} \end{pmatrix} \\ &= -i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\begin{pmatrix} [\sqrt{p\cdot\sigma}\,(-i\sigma^2)]_{as'} \\ -[\sqrt{p\cdot\bar\sigma}\,(-i\sigma^2)]_{as'} \end{pmatrix} \\ &= -\gamma^1\gamma^3 v^{s'}(\vec p)\;. \end{aligned} \tag{202}$$

With Eqs. (200) and (202), Eq. (190) becomes

$$\begin{aligned} &\mathcal{T}^{-1}\psi(\vec x,t)\,\mathcal{T} = -\gamma^1\gamma^3\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec p}}}\times \\ &\sum_s\left\{\eta_t^* a_s(\vec p)\,u^s(\vec p)\,e^{-ip_\mu\tilde x^\mu_t} + \bar\eta_t b^\dagger_s(\vec p)\,v^s(\vec p)\,e^{ip_\mu\tilde x^\mu_t}\right\}\;. \end{aligned} \tag{203}$$

Following our discussion of parity in Eqs. (158) and (159), in a time-reversal-invariant theory we must require that $\mathcal{T}^{-1}\psi(\vec x,t)\,\mathcal{T}$ anticommute with $\bar\psi(x)$ at spacelike separations, which in turn requires the ratio of coefficients of the creation and annihilation pieces of Eq. (203) to be 1. So

$$\eta_t\bar\eta_t = 1\;, \tag{204}$$

and

$$\mathcal{T}^{-1}\psi(\vec x,t)\,\mathcal{T} = -\eta_t^*\gamma^1\gamma^3\psi(\vec x,-t)\;. \tag{205}$$

Time Reversal and Dirac Field Bilinears

Again it is fairly straightforward to calculate the effect of $\mathcal{T}$ on the Dirac bilinears, so I will give only the results, in a format that matches the description of parity transformations in Eq. (161):

$$\begin{array}{ll}
\mathcal{T}^{-1}\bar{\psi}\psi\,\mathcal{T} = +\mathcal{O}(\vec{x},-t) & \mathcal{T}^{-1}i\bar{\psi}\gamma^5\psi\,\mathcal{T} = -\mathcal{O}(\vec{x},-t) \\
\mathcal{T}^{-1}\bar{\psi}\gamma^i\psi\,\mathcal{T} = -\mathcal{O}(\vec{x},-t) & \mathcal{T}^{-1}\bar{\psi}\gamma^i\gamma^5\psi\,\mathcal{T} = -\mathcal{O}(\vec{x},-t) \\
\mathcal{T}^{-1}\bar{\psi}\gamma^0\psi\,\mathcal{T} = +\mathcal{O}(\vec{x},-t) & \mathcal{T}^{-1}\bar{\psi}\gamma^0\gamma^5\psi\,\mathcal{T} = +\mathcal{O}(\vec{x},-t) \\
\mathcal{T}^{-1}i\bar{\psi}\left[\gamma^0,\gamma^i\right]\psi\,\mathcal{T} = +\mathcal{O}(\vec{x},-t) & \mathcal{T}^{-1}i\bar{\psi}\left[\gamma^i,\gamma^j\right]\psi\,\mathcal{T} = -\mathcal{O}(\vec{x},-t)
\end{array} \tag{206}$$

where the quantity $\mathcal{O}$ on the right-hand side of each equation denotes the quantity between $\mathcal{T}^{-1}$ and $\mathcal{T}$ on the left.

One can always choose $\eta_t = \bar{\eta}_t = 1$, since the phase of an antilinear operator, such as $\mathcal{T}$, changes if one rotates that phases of all vectors in the Hilbert space.

Charge Conjugation: Particle ⟷ Antiparticle

Having done the other cases, charge conjugation C is very straightforward.

C maps particle states into antiparticle states and vice versa, and it commutes with all the generators of the Poincaré group. Labeling particle and antiparticle states with subscripts a and b respectively,

$$\begin{aligned} C\,|\vec{p},s\rangle_a &= \eta_C\,|\vec{p},s\rangle_b \\ C\,|\vec{p},s\rangle_b &= \bar{\eta}_C\,|\vec{p},s\rangle_a \ . \end{aligned} \tag{207}$$

The corresponding relations for creation and annihilation operators are given by

$$\begin{aligned} C^{-1}a_s^\dagger(\vec{p})\,C &= \bar{\eta}_C^* b_s^\dagger(\vec{p})\,, \quad C^{-1}a_s(\vec{p})\,C = \bar{\eta}_C b_s(\vec{p})\ , \\ C^{-1}b_s^\dagger(\vec{p})\,C &= \eta_C^* a_s^\dagger(\vec{p})\,, \quad C^{-1}b_s(\vec{p})\,C = \eta_C a_s(\vec{p})\ . \end{aligned} \tag{208}$$

C and the Dirac Field:

Using again

$$\psi_a(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left\{ a_s(\vec{p})\, u_a^s(\vec{p})\, e^{-ip_\mu x^\mu} + b_s^\dagger(\vec{p})\, v_a^s(\vec{p})\, e^{ip_\mu x^\mu} \right\} .$$

we see that

$$C^{-1}\psi_a(x)\, C = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \times \sum_s \left\{ \bar{\eta}_C b_s(\vec{p})\, u_a^s(\vec{p})\, e^{-ip_\mu x^\mu} + \eta_C^* a_s^\dagger(\vec{p})\, v_a^s(\vec{p})\, e^{ip_\mu x^\mu} \right\} . \tag{209}$$

Since this expression involves $a_s^\dagger(\vec{p})$ and $b_s(\vec{p})$, it will have to be compared with $\psi^\dagger(x)$ and not $\psi(x)$. This means that we will need a relation between the u's and the v's.

Relation Between the u's and the v's:

$$\begin{aligned} u^{s*}(\vec{p}) &= \begin{pmatrix} u_{La}^{s*}(\vec{p}) \\ u_{Ra}^{s*}(\vec{p}) \end{pmatrix} \\ &= \begin{pmatrix} (\sqrt{p\cdot\sigma})_{as}^* \\ (\sqrt{p\cdot\bar{\sigma}})_{as}^* \end{pmatrix} \\ &= \begin{pmatrix} [\sigma^2(\sqrt{p\cdot\bar{\sigma}})\sigma^2]_{as} \\ [\sigma^2(\sqrt{p\cdot\sigma})\sigma^2]_{as} \end{pmatrix} \\ &= -i\left[\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}(-i\sigma^2) \\ -\sqrt{p\cdot\bar{\sigma}}(-i\sigma^2) \end{pmatrix} \right]_{as} \\ &= -i\gamma^2 v^s(\vec{p}) . \end{aligned} \tag{210}$$

The complex conjugate of this equation implies that

$$v^{s*}(\vec{p}) = -i\gamma^2 u^s(\vec{p}) . \tag{211}$$

C and the Dirac Field (cont):

Using Eqs. (210) and (211), Eq. (209) becomes

$$C^{-1}\psi(x)\,C = -i\gamma^2 \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec p}}} \times \left[\sum_s \left\{ \bar\eta_C^* b_s^\dagger(\vec p)\, v_a^s(\vec p)\, e^{ip_\mu x^\mu} + \eta_C a_s(\vec p)\, u_a^s(\vec p)\, e^{-ip_\mu x^\mu} \right\}\right]^\dagger . \tag{212}$$

Causality in this case requires that if C is a valid symmetry, then

$$\eta_c \bar\eta_C = 1\ , \tag{213}$$

and then

$$C^{-1}\psi_a(x)\,C = -i\eta_C\left(\psi^\dagger(x)\gamma^2\right)_a^T = -i\eta_C\left(\bar\psi(x)\gamma^0\gamma^2\right)_a^T\ , \tag{214}$$

where the superscript T denotes the transpose. Note that one can choose to define C so that $\eta_C = \bar\eta_C = 1$.

C and the Dirac Field Bilinears:

The charge conjugation of Dirac field bilinears introduces one new complication, so we will work out one case in detail: $C^{-1}\bar\psi\gamma^0\psi\,C$. To understand the physical significance of this operator, note that

$$j^\mu \equiv \bar\psi\gamma^\mu\psi \tag{215}$$

is a conserved current, since the Dirac equation in its original (Eq. (110)) and adjoint (Eq. (122)) form implies that

$$\gamma^\mu\,\partial_\mu\psi = -im\psi\,, \quad \bar\psi\,\gamma^\mu\,\overleftarrow{\partial_\mu} = im\bar\psi\ .$$

Therefore

$$\partial_\mu\left(\bar\psi\gamma^\mu\psi\right) = \bar\psi\gamma^\mu\left(\overleftarrow{\partial_\mu} + \overrightarrow{\partial_\mu}\right)\psi = 0\ . \tag{216}$$

(It is worth noting that an analogous calculation shows that $\partial_\mu\left(\bar\psi\gamma^\mu\gamma^5\psi\right) = 2im\bar\psi\gamma^5\psi$, so the axial vector current can be conserved only in the limit $m \to 0$.) Since $j^\mu(x)$ is conserved, it is identified with the electromagnetic current, up to a constant factor.

To understand the normalization of $j^\mu(x)$, we can express the conserved charge as an operator. Write $\psi_a(\vec{x},0)$ as

$$\psi_a(\vec{x},0) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left\{ a_s(\vec{p})\, u_a^s(\vec{p}) + b_s^\dagger(-\vec{p})\, v_a^s(-\vec{p}) \right\} e^{i\vec{p}\cdot\vec{x}} \ ,$$

and then

$$\begin{aligned} Q &\equiv \int \mathrm{d}^3x\, j^0(\vec{x},0) = \int \mathrm{d}^3x\, \psi^\dagger(\vec{x},0)\, \psi(\vec{x},0) \\ &= \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \int \frac{\mathrm{d}^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{q}}}} \times \\ &\quad \times \sum_{rs} \left\{ a_s^\dagger(\vec{p})\, \bar{u}^s(\vec{p}) + b_s(-\vec{p})\, \bar{v}^s(-\vec{p}) \right\} e^{-i\vec{p}\cdot\vec{x}} \\ &\quad \times \left\{ a_r(\vec{q}) u^r(\vec{q}) + b_r^\dagger(-\vec{q})\, v^r(-\vec{q}) \right\} e^{i\vec{q}\cdot\vec{x}} \ . \end{aligned}$$

Integration over $\vec{x}$ gives a factor $(2\pi)^3\, \delta^{(3)}(\vec{p}-\vec{q})$, which can then be used to integrate over $\vec{q}$. Then one uses the relations

$$\begin{array}{ll} u_r^\dagger(\vec{p})\, u_s(\vec{p}) = 2E_{\vec{p}}\, \delta_{rs} & v_r^\dagger(\vec{p})\, v_s(\vec{p}) = 2E_{\vec{p}}\, \delta_{rs} \\ u_r^\dagger(\vec{p})\, v_s(-\vec{p}) = 0 & v_r^\dagger(\vec{p})\, u_s(-\vec{p}) = 0 \end{array} \qquad (217)$$

to obtain

$$Q = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_s \left\{ a_s^\dagger(\vec{p})\, a_s(\vec{p}) + b_s(\vec{p})\, b_s^\dagger(\vec{p}) \right\} \ .$$

One sees that this contains an infinite vacuum contribution, since

$$\sum_s b_s(\vec{p})\, b_s^\dagger(\vec{p}) = 2(2\pi)^3\, \delta^{(3)}(0) - \sum_s b_s^\dagger(\vec{p})\, b_s(\vec{p}) \ .$$

Thus we can write

$$Q = \,:\!Q\!:\, + Q_{\rm vac} \;, \tag{218}$$

where

$$Q = \int \frac{d^3p}{(2\pi)^3} \sum_s \left\{ a_s^\dagger(\vec p)\, a_s(\vec p) - b_s^\dagger(\vec p)\, b_s(\vec p) \right\} \;, \tag{219}$$

and

$$Q_{\rm vac} = 2 \int \frac{d^3p}{(2\pi)^3} \times \text{Volume of space} \;, \tag{220}$$

analogous to Eq. (135) for the energy. Here I am using

$$\delta^{(3)}(0) = \delta^{(3)}\,(\vec p)\big|_{\vec p=0} = \int \frac{d^3x}{(2\pi)^3}\, e^{i\vec p\cdot\vec x}\Big|_{\vec p=0} = \frac{1}{(2\pi)^3} \times \text{Volume of space} \;.$$

From Eq. (219) one can see that Q counts each particle as one unit and each antiparticle as minus one unit, so the electromagnetic charge is given by Qe, where e is the charge of an electron ($e < 0$). Note that $Q_{\rm vac}$, as given by Eq. (220), can be interpreted as the charge of the particles that are needed to fill the Dirac sea, as described on slide 22 of the 5/9/06 lecture. Note that the particles filling the Dirac sea have the same sign charge as the normal particles. It is the **holes** that have the opposite charge.

Now that we understand j^0 and its integral Q, we can apply C, using Eq. (214) and its adjoint,

$$C^{-1} \psi_a^\dagger(x)\, C = -i\eta_C^* \left(\gamma^2 \psi(x)\right)_a \;. \tag{221}$$

Then

$$\begin{aligned} C^{-1}\psi_a^\dagger\psi_a C &= C^{-1}\psi_a^\dagger C\, C^{-1}\psi_a C \\ &= -i\eta_C^*\left(\gamma^2\psi\right)_a\left(-i\eta_C\right)\left(\psi^\dagger\gamma^2\right)_a^T \\ &= -\gamma_{ab}^2\psi_b\,\psi_c^\dagger\gamma_{ca}^2 = \psi_a\,\psi_a^\dagger \\ &= -\psi_a^\dagger\,\psi_a + 4\delta^{(3)}(0)\ . \end{aligned} \tag{222}$$

The $\delta^{(3)}(0)$ piece looks odd, but note that it would disappear if we defined normal products which subtracted the divergent vacuum contributions at the start. That is, Eq. (218) corresponds to an equation for densities

$$j^0 = \psi^\dagger\psi = \,:\psi^\dagger\psi:\, + 2\delta^{(3)}(0)\,, \text{ so } :\psi^\dagger\psi: \,= \psi_a^\dagger\psi_a - 2\delta^{(3)}(0)\ .$$

Then, Eq. (222) can be rewritten as

$$\begin{aligned} C^{-1} :\psi_a^\dagger\psi_a: C &= C^{-1}\psi_a^\dagger\psi_a C - 2\delta^{(3)}(0) \\ &= -\psi_a^\dagger\psi_a + 2\delta^{(3)}(0) \\ &= -:\psi^\dagger\psi:\ . \end{aligned} \tag{223}$$

Thus, if C is applied to the bare charge density operator, $\psi^\dagger\psi$, it changes the sign of the vacuum contribution as well as the contribution of the physical particles, and thus Eq. (222) contains a divergent term. However, if the divergent vacuum contribution is removed by normal ordering, as in Eq. (223), conjugation by C does not reintroduce any infinite quantities.

The fact that C will not reintroduce infinities is easy to understand from a different point of view. If one looks back at Eq. (208), one sees that C interchanges particles and antiparticles, but it does not interchange creation and annihilation operators. Thus, if any expression is written in explicitly normal ordered form, with all annihilation operators written to the right of all creation operators, then conjugation with C will leave it in explicitly normal ordered form.

Finally, I list a table showing the effect of charge conjugation on all the Dirac field bilinears. We assume that all the operators have been normal-ordered, so that infinite terms will not appear.

$$\begin{array}{ll} C^{-1}\bar\psi\psi\, C = +\mathcal{O}(x) & C^{-1}i\bar\psi\gamma^5\psi\, C = +\mathcal{O}(x) \\ C^{-1}\bar\psi\gamma^i\psi\, C = -\mathcal{O}(x) & C^{-1}\bar\psi\gamma^i\gamma^5\psi\, C = +\mathcal{O}(x) \\ C^{-1}\bar\psi\gamma^0\psi\, C = -\mathcal{O}(x) & C^{-1}\bar\psi\gamma^0\gamma^5\psi\, C = +\mathcal{O}(x) \\ C^{-1}i\bar\psi\left[\gamma^0\,,\,\gamma^i\right]\psi\, C = -\mathcal{O}(x) & C^{-1}i\bar\psi\left[\gamma^i\,,\,\gamma^j\right]\psi\, C = -\mathcal{O}(x) \end{array} \tag{224}$$

where the quantity $\mathcal{O}$ on the right-hand side of each equation denotes the quantity between C^{-1} and C on the left.

Consequences of Discrete Symmetries

☆ C and $\mathcal{P}$: For spin-$\frac{1}{2}$ particles, the parity of antiparticles is opposite of the parity of particles $\Longrightarrow$ ground state of positronium has odd $\mathcal{P}$; C is odd for $S = 1$ positronium (interchanges 2 fermions in symmetric state), and even for $S = 0$ positronium (odd wavefunction). Implies $S = 0$ can decay to two photons, but not $S = 1$. (See *Advanced Quantum Mechanics,* by J.J. Sakurai, Sec. 4-4.)

☆ $\mathcal{T}^2 = -1$ for states with odd numbers of spin-$\frac{1}{2}$ particles $\Longrightarrow$ "Kramers degeneracy". Even if rotational symmetry is completely broken, say by a static $\vec E$ field ($\mathcal{T}$-preserving, unlike $\vec B$), $|\psi\rangle$ and $\mathcal{T}\,|\psi\rangle$ must be **distinct** states with the same energy. (See *The Quantum Theory of Fields,* Vol. 1, by Steven Weinberg, p. 81.)

☆ $C\mathcal{T}\mathcal{P}$ symmetry: implies that a thermal equibrium state always has baryon number zero. Baryogenesis in early universe therefore requires a departure from thermal equilibrium. (See *The Early Universe,* by Edward W. Kolb and Michael S. Turner, Chapter 6.)