Chapter 8

Meson Mass Matrix

Close up on meson mass matrix states as:

$$
\begin{aligned}
V_m &= \upsilon\{(\frac{1}{F}\frac{a}{\sqrt{3}} + \frac{1}{8}\frac{\pi^\circ}{\sqrt{2}} + \frac{1}{f}\frac{n}{\sqrt{6}})^2 m_u + \\
&\quad (\frac{1}{F}\frac{a}{\sqrt{3}} - \frac{1}{8}\frac{\pi^\circ}{\sqrt{2}} + \frac{1}{f}\frac{n}{\sqrt{6}})^2 m_d + \\
&\quad (\frac{1}{F}\frac{a}{\sqrt{3}} - \frac{1}{f}\frac{2n}{\sqrt{6}})^2 m_s\} \qquad (8.1)\\
&= \frac{\upsilon}{2f^2}\begin{pmatrix} a & \pi^\circ & \eta \end{pmatrix} \\
&\quad \begin{pmatrix} \frac{f^2}{F^2}\frac{2}{3}(m_u + m_d + m_s) & \frac{2f}{F}\frac{m_u - m_d}{\sqrt{6}} & \frac{2f}{F}\frac{m_u + m_d - 2m_s}{3\sqrt{2}} \\ \frac{2f}{F}\frac{m_u - m_d}{\sqrt{6}} & (m_u + m_d) & \frac{1}{\sqrt{3}}(m_u - m_d) \\ \frac{2f}{F}\frac{m_u + m_d - 2m_s}{3\sqrt{2}} & \frac{1}{\sqrt{3}}(m_u - m_d) & \frac{1}{3}(m_u + m_d + 4m_s) \end{pmatrix} \\
&\quad \begin{pmatrix} a \\ \pi^\circ \\ \eta \end{pmatrix} \qquad (8.2)
\end{aligned}
$$

So

$$
\frac{\mathcal{M}^\in}{(\frac{\upsilon}{f^2})} = \begin{pmatrix} \frac{f^2}{F^2}\frac{2}{3}(m_u + m_d + m_s) & \frac{2f}{F}\frac{m_u - m_d}{\sqrt{6}} & \frac{2f}{F}\frac{m_u + m_d - 2m_s}{3\sqrt{2}} \\ \frac{2f}{F}\frac{m_u - m_d}{\sqrt{6}} & (m_u + m_d) & \frac{1}{\sqrt{3}}(m_u - m_d) \\ \frac{2f}{F}\frac{m_u + m_d - 2m_s}{3\sqrt{2}} & \frac{1}{\sqrt{3}}(m_u - m_d) & \frac{1}{3}(m_u + m_d + 4m_s) \end{pmatrix} \qquad (8.3)
$$

1. For $\frac{f}{F} \ll 1$ (axions). There is a small eigenvalue $\sim \frac{f^2}{F^2}$ associated with an eigenvalue $\sim \begin{pmatrix} 1 \\ \frac{f}{F}a \\ \frac{f}{F}b \end{pmatrix}$ with

$$2\frac{m_u - m_d}{\sqrt{6}} + (m_u + m_d)a + \frac{1}{\sqrt{3}}(m_u - m_d)b = 0 \quad (8.4)$$

$$\frac{\sqrt{2}}{3}(m_u + m_d - 2m_s) + \frac{1}{\sqrt{3}}(m_u - m_d)a + \frac{1}{3}(m_u + m_d + 4m_s)b = 0 \quad (8.5)$$

After some algebra we find

$$a = -\sqrt{\frac{3}{2}}\frac{m_s(m_u - m_d)}{m_u m_d + m_u m_s + m_d m_s} \quad (8.6)$$

$$b = \frac{1}{\sqrt{2}}\frac{m_u m_s + m_d m_s - 2m_u m_d}{m_u m_d + m_u m_s + m_d m_s} \quad (8.7)$$

and $(mass)^2$

$$\mu^2 = \frac{3m_u m_d m_s}{m_u m_d + m_u m_s + m_d m_s}\frac{\upsilon}{F^2} \quad (8.8)$$

$$\simeq \frac{3m_u m_d m_s}{m_u m_d + m_u m_s + m_d m_s}\frac{2}{m_u + m_d}\frac{f^2}{F^2}m_\pi^2 \quad (8.9)$$

2. In opposite limit $\frac{f}{F} \gg 1$, note that if we take $m_u = m_d = 0$

$$\frac{\mathcal{M}^{\in}}{(\frac{\upsilon}{f^2})} = \begin{pmatrix} \frac{f^2}{F^2}\frac{2}{3}m_s & 0 & \frac{f}{F}\frac{-4}{3\sqrt{2}}m_s \\ 0 & 0 & 0 \\ \frac{f}{F}\frac{-4}{3\sqrt{2}}m_s & 0 & \frac{4}{3}m_s \end{pmatrix} \quad (8.10)$$

has two vanishing eigenvalues. So η gets infected and the GM-O relation is badly violated. The general case is a little messy but with $m_u = m_d \ll m_s$ we easily arrive at

$$m^2_{infected\ \eta} \to \frac{\upsilon}{f^2}3\frac{(m_u + m_d)m_s}{m_u + m_d + m_s} = 3m_\pi^2 \quad (8.11)$$

In general, this serves as a bound for all values of $\frac{F}{f}$. The low light meson is unacceptable unless it is very weakly coupled. This is the $U_A(1)$ problem.