8.324 Relativistic Quantum Field Theory II

1.3.3: Field equations and conservation laws

We begin with the Lagrangian we considered in the last lecture:

$$\begin{aligned}\mathscr{L} &= -\frac{c}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) - i\bar{\Psi}(\gamma^\mu D_\mu - m)\Psi \\ &= -\frac{c}{4}\mathrm{Tr}(F^a_{\mu\nu}F^{\mu\nu b}T^aT^b) - i\bar{\Psi}(\gamma^\mu(\partial_\mu - igA^a_\mu T^a) - m)\Psi. \end{aligned} \tag{1}$$

For the gauge group $SU(n)$, we can choose T^a such that $\mathrm{Tr}(T^aT^b) = \delta_{ab}$, and so $c = 1$. We now obtain the equations of motion.

A. Varying A_μ :

$$\delta F^a_{\mu\nu} = \partial_\mu \delta A^a_\nu - \partial_\nu \delta A^a_\mu + gf^a_{bc}(\delta A^b_\mu A^c_\nu + A^b_\mu \delta A^c_\nu) = (D_\mu \delta A_\nu)^a - (D_v \delta A_\mu)^a \tag{2}$$

where $(D_\mu \delta A_\nu)^a \equiv \partial_\mu \delta A^a_\nu + gf^a_{bc}A^b_\mu \delta A^c_\nu$. Now,

$$\delta\mathscr{L} = -\frac{1}{2}\delta F^a_{\mu\nu}F^{\mu\nu a} - g\bar{\Psi}\gamma^\mu T^a\Psi\delta A^a_\mu, \tag{3}$$

and so

$$(D_\mu F^{\mu\nu})^a = J^{\nu a}, \tag{4}$$

where $(D_\mu F^{\mu\nu})^a \equiv \partial_\mu F^{\mu\nu a} + gf^a_{bc}A^b_\mu F^{\mu\nu c}$, and $J^{\nu a} \equiv g\bar{\Psi}\gamma^\nu T^a\Psi$. Equation (4) can also be written as:

$$\partial_\mu F^{\mu\nu a} = j^{\nu a} \tag{5}$$

with $j^{\nu a} \equiv g\bar{\Psi}\gamma^\nu T^a\Psi - gf^a_{bc}A^b_\mu F^{\mu\nu c}$.

B. Varying Ψ :

$$(\gamma^\mu D_\mu - m)\Psi = 0. \tag{6}$$

We note for emphasis that this is a matrix equation. Now, we recall that in quantum electrodynamics, $\partial_\mu F^{\mu\nu} = j^\nu$, with $j^\nu \equiv e\bar{\Psi}\gamma^\nu\Psi$ and $\partial_\nu j^\nu = 0$, that is, j^ν is a conserved current. When $j = 0$, A_μ is a free field, and we obtain free electromagnetic wave solutions.

Remarks:

1. In the non-Abelian case, the theory for A_μ remains interacting with $\Psi = 0$. In quantum electrodynamics, A_μ is neutral, whereas, in the non-Abelian case, A^a_μ carries the group index, and so is charged under itself, leading to self-interaction.

2. In terms of $F_{\mu\nu} = F^a_{\mu\nu}T^a$ and $J^\nu = J^{\nu a}T^a$, we have from (4) that

$$D_\mu F^{\mu\nu} = J^\nu, \tag{7}$$

with $D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} - ig\,[A_\mu, F^{\mu\nu}]$, and under a gauge transformation,

$$\begin{aligned} F_{\mu\nu} &\longrightarrow VF_{\mu\nu}V^\dagger, \\ D_\mu F^{\mu\nu} &\longrightarrow VD_\mu F^{\mu\nu}V^\dagger. \end{aligned} \tag{8}$$

more generally, for any $X = X^aT^a$, which transforms as $X \longrightarrow VXV^\dagger$, $D_\mu X = \partial_\mu X - ig\,[A_\mu, X]$ transforms as $D_\mu X \longrightarrow V(D_\mu X)V^\dagger$. From (7), we have that

$$J^\nu \longrightarrow VJ^\nu V^\dagger. \tag{9}$$

This will be checked directly in the problem sets.

3. Acting with D_ν on (7),

$$D_\nu D_\mu F^{\mu\nu} = \frac{1}{2}\left[D_\nu, D_\mu\right] F^{\mu\nu} = \frac{1}{2}\left[F_{\nu\mu}, F^{\mu\nu}\right] = 0,$$

and so $D_\nu J^\nu = 0$. This can also be checked directly from the equations of motion.

4. J^ν is precisely the conserved Noether current for global $SU(n)$ symmetry in the absence of A_μ. For $A_\mu \neq 0$, J^ν is covariantly conserved. Of course, (1) is also invariant under global transformations

$$\Psi(x) \longrightarrow V\Psi(x),\ A_\mu(x) \longrightarrow VA_\mu(x)V^\dagger, \tag{10}$$

with V a position-independent $SU(n)$ matrix. The Noether current for this global symmetry is precisely j^ν, which was introduced in (5). From (5),

$$\partial_\mu j^\mu = 0. \tag{11}$$

Note that j^ν depends on A_μ non-trivially, which is true if and only if A^a_μ is charged. j^ν is not gauge invariant: it does not have good transformation properties.

1.3.4 Further generalizations

A **representation of a Lie group** G on a vector space V is a linear action

$$g \cdot v \in V \tag{12}$$

for $g \in G$, $v \in V$, such that

$$g_1 \cdot (g_2 \cdot v) = (g_1 \circ g_2) \cdot v \tag{13}$$

where $\circ$ is the group product. Here, V is the **representation space**.

A **representation of a Lie algebra** $\mathfrak{g}$ on a vector space V is a linear action

$$x \cdot v \in V \tag{14}$$

for $g \in G$, $v \in V$, such that

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v) \tag{15}$$

where $[\,,\,]$ is defined using the group product. The concept of a representation is, again, tailor-made for physics. Here, V is the physical space, and the same abstract group can appear in different physical contexts, with different V (with different representations). We note that a representation for G induces a representation for $\mathfrak{g}$, and visa versa.

Example 1: Angular momentum, $SU(2)$

A spin-j representation is described by $(2j+1)\times(2j+1)$ matrices, acting on a $(2j+1)$-dimensional V. For $SU(n)$, representing the group as $n \times n$ unitary matrices acting on n-dimensional complex vectors gives the **fundamental representation.** Here,

$$A_\mu = A^a_\mu T^a \in \mathfrak{g}, \tag{16}$$

with T^a in the fundamental representation. More generally, a representation r of $\mathfrak{g}$ (or G) of dimension d_r is defined by $d_r \times d_r$ matrices $T^{(r)}_a$ that represent the generators, satisfying

$$\left[T^{(r)}_a, T^{(r)}_b\right] = i f^c_{ab} T^{(r)}_c. \tag{17}$$

One can prove that, for compact groups,

$$\mathrm{Tr}(T^{(r)}_a T^{(r)}_b) = C(r)\delta_{ab}, \tag{18}$$

where $C(r)$ is a positive number depending on the representation r. For non-compact groups, $\mathrm{Tr}(T^{(r)}_a T^{(r)}_b)$ is not positive-definite. Amongst all representations, there is a special one for all G (and $\mathfrak{g}$). The **adjoint representation**, which is universal for all Lie groups and Lie algebras, has as its representation space the vector space for the Lie algebra itself. That is, $V = \mathfrak{g}$ ($\dim V = \dim \mathfrak{g}$). The action of the Lie algebra is defined, for $x \in V = \mathfrak{g}$, $y \in \mathfrak{g}$, by

$$y \cdot x \equiv [y, x]\,. \tag{19}$$

We need to show that this rule satisfies (15), that is,

$$y_1 \cdot (y_2 \cdot x) - y_2 \cdot (y_1 \cdot x) = [y_1, y_2] \cdot x. \tag{20}$$

The left-hand side of this equation is $[y_1, [y_2, x]] - [y_2, [y_1, x]]$, and the right-hand side is $[[y_1, y_2], x]$. The equality of these follows from the Jacobi identity, which is an automatic consequence of the associativity of the group product, so (19) indeed gives a representation. For $SU(n)$, the $T_a^{(adj)}$ are $(n^2-1)\times(n^2-1)$ matrices. Now we go back to the theory, and generalize as follows: we consider a general compact group G in place of $SU(n)$, with Ψ in the vector space of some representation r of G. $A_\mu = A_\mu^a T_a^{(r)} \in \mathfrak{g}$. Now we take

$$\mathscr{L} = -\frac{c}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) - i\bar{\Psi}(\gamma^\mu D_\mu - m)\Psi, \tag{21}$$

with $F_{\mu\nu} \equiv F_{\mu\nu}^a T_a^{(r)}$, $D_\mu \equiv \partial_\mu - igA_\mu^a T_a^{(r)}$. Since $\mathrm{Tr}(T_a^{(r)}T_b^{(r)}) = C(r)\delta_{ab}$, to maintain canonical normalization, $c = \frac{1}{C(r)}$. Here, compactness of G is essential, as non-compactness leads to the wrong sign for kinetic terms of A_μ^a. Now, the action of the Lie group G on $V = \mathfrak{g}$ is given, for $g \in G$, $x \in \mathfrak{g}$, by

$$g \cdot x \equiv g \circ x \circ g^{-1}. \tag{22}$$

It is easy to check that this satisfies the group action product rule (13). For infinitesimal $g = 1 + iy$, $y \in \mathfrak{g}$, this action reduces to (19). In matrix form, for $x \in V = \mathfrak{g}$, $x = x^b T_b$ (the upper and lower indices can be distinct for this general treatment)

$$T_a^{(adj)} \cdot x = [T_a, x] = [T_a, T_b]\,x^b = if^c_{\ ab}T_c x^b. \tag{23}$$

On the other hand,

$$T_a^{(adj)} \cdot x = (T_a^{(adj)} \cdot x)^c T_c \tag{24}$$

where $(T_a^{(adj)}x)^c \equiv (T_a^{(adj)})^c_{\ b}x^b$. Hence,

$$(T_a^{(adj)})^c_{\ b} = if^c_{\ ab}. \tag{25}$$

Remarks:

1. Consider M^a in the adjoint representation, $a = 1, \ldots, \dim \mathfrak{g}$;

$$\begin{aligned}(D_\mu M)^a &= \partial_\mu M^a - igA_\mu^a (T_b^{(adj)} \cdot M)^a \\ &= \partial_\mu M^a + gf^a_{\ bc}A_\mu^b M^c,\end{aligned}$$

where the second line follows from (25). In matrix form, with $M = M^a T_a^{(r)}$ for any representation r,

$$\begin{aligned}D_\mu M &= \partial_\mu M - igA_\mu^a T_a^{(adj)} \cdot M \\ &= \partial_\mu M - igA_\mu^a T^{(r)} \\ &= \partial_\mu M - ig\,[A_\mu, M]\end{aligned}$$

with $A_\mu = A_\mu^a T_a$.

2. Under a gauge transformation,

$$\begin{aligned}M &\longrightarrow VMV^{-1}, \\ M^a T_a &\longrightarrow M^a V T_a V^{-1},\end{aligned}$$

and hence,

$$M^a \longrightarrow M^a D_a^{\ b}, \tag{26}$$

where $VT_aV^{-1} \equiv D_a^{\ b}T_b$.