8.324 Relativistic Quantum Field Theory II

Lecture 8

In the last lecture, we showed, for a general interacting theory:

$$G_F(p) = \frac{-iZ}{p^2 + m^2 - i\epsilon} + \int_{4m^2}^{\infty} d\mu^2\, \sigma(\mu^2) \frac{i}{p^2 + \mu^2 - i\epsilon}, \tag{1}$$

where the first term is the contribution from single-particle states, and the second term is the contribution from multi-particle states. From this, we have that

$$\mathfrak{Im}(iG_F(p)) = \sum_i \pi\delta(p^2 + m_i^2) Z_i + \pi\sigma(-p^2). \tag{2}$$

This is the spectral function, picking out the physical on-shell states. There is one more sum rule we wish to observe. Begin with the canonical quantization:

$$\left[\dot{\phi}(t, \vec{x}), \phi(t, \vec{y})\right] = -i\delta(\vec{x} - \vec{y}). \tag{3}$$

As this operator is just a complex number, we can equate it with its expectation value:

$$\begin{aligned}
\langle 0| \left[\dot{\phi}(t, \vec{x}), \phi(t, \vec{y})\right] |0\rangle =& -i\delta(\vec{x} - \vec{y}) \\
=& \partial_t \langle 0| \phi(t, \vec{x})\phi(t', \vec{y}) |0\rangle|_{t'\to t} - \partial_t \langle 0| \phi(t', \vec{y})\phi(t, \vec{x}) |0\rangle|_{t'\to t} \\
=& \partial_t G_+(x - y)|_{t'\to t} - \partial_t G_+(y - x)|_{t'\to t} \\
=& \int_0^{\infty} d\mu^2\, \rho(\mu^2) \left[\partial_t G_+^{(0)}(x - y;\, \mu^2)\Big|_{t'\to t} - \partial_t G_+^{(0)}(y - x;\, \mu^2)\Big|_{t'\to t}\right] \\
=& 2\int_0^{\infty} d\mu^2\, \rho(\mu^2)\; \partial_t G_+^{(0)}(x - y;\, \mu^2)\Big|_{t'\to t}.
\end{aligned}$$

Recall, in the free theory, we have that

$$\partial_t\, G_+^{(0)}(x - y)\Big|_{t'\to t} = -\frac{i}{2}\delta(\vec{x} - \vec{y}), \tag{4}$$

and so, we have that

$$1 = \int_0^{\infty} d\mu^2\, \rho(\mu^2). \tag{5}$$

Because $\rho(\mu^2) = \sigma(\mu^2) + \sum_i Z_i\delta(\mu^2 - m_i^2)$, where both terms are greater than or equal to zero for all values of μ^2, we have that

$$\int_{4m^2}^{\infty} d\mu^2\, \sigma(\mu^2) < 1,\;\; Z_i < 1. \tag{6}$$

In reality, the above argument may not always hold due to possible ultraviolet divergences. The same discussion can be generalized to a spinor, as seen in the problem set, and to a vector, as we will discuss later.

2.2: AN EXPLICIT EXAMPLE

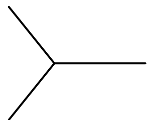
Consider a scalar Lagrangian with a ϕ^3 interaction:

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 + \frac{1}{6}g\phi^3. \tag{7}$$

This leads to the Feynman rules:

p

$$= \frac{-i}{p^2 + m^2 - i\epsilon},$$

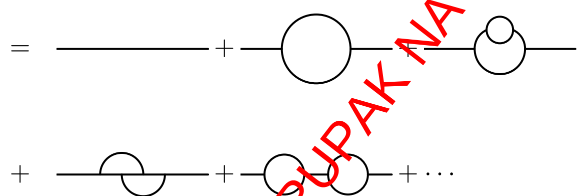


$$= ig, \tag{8}$$

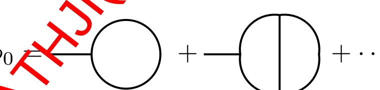
and by dimensional analysis, we find

$$[\phi] = \frac{d-2}{2},$$
$$[x] = -1,$$
$$[g] = 3 - \frac{d}{2},$$

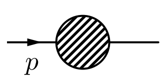
where we work in a general space-time dimension d. We note that g is dimensionless in $d = 6$.

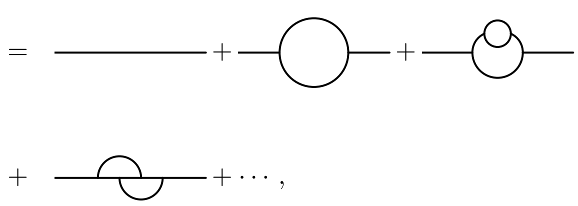
$G_F(p) =$  (9)

Note that we do not consider diagrams involving tadpoles,

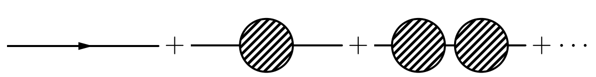
$\langle 0|\,\phi\,|0\rangle \equiv \phi_0 =$  (10)

as we can always shift the field definition to $\phi = \phi_0 + \tilde{\phi}$, so that $\langle 0|\,\tilde{\phi}\,|0\rangle = 0$, and the sum of the tadpole sub-diagrams gives zero. Now, we define 1PI, or one-particle irreducible diagrams, as the diagrams which cannot be separated into two disconnected parts by cutting one propagator. We denote the sum of 1PI diagrams by

$i\Pi(p) =$ 
p

$=$  (11)

and so, we have that

$G_F(p) =$  (12)

That is,

$$\begin{aligned}
G_F(p) =& G_F^{(0)} + G_F^{(0)} i\Pi G_F^{(0)} + G_F^{(0)} i\Pi G_F^{(0)} i\Pi G_F^{(0)} + \cdots \\
=& G_F^{(0)} \frac{1}{1 - i\Pi G_F^{(0)}} \\
=& \frac{1}{\left(G_F^{(0)}\right)^{-1} - i\Pi} \\
=& \frac{-i}{p^2 + m_0^2 - \Pi(p) - i\epsilon}
\end{aligned}$$

where $\Pi(p)$ is the self-energy. We note that Π is, in fact, a function of p^2 only, by Lorentz symmetry. Before evaluating $\Pi(p^2)$ explicitly to lowest order, we make two remarks:

1. The physical mass, the pole of $G_F(p)$, is given by $p^2 + m_0^2 - \Pi(p^2) = 0$. That is, the physical mass m^2 satisfies

$$m^2 - m_0^2 + \Pi(-m^2) = 0 \tag{13}$$

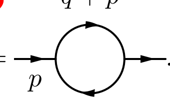
2. The field renormalization Z, the residue of the pole, is given by expanding around the pole to lowest order:

$$\begin{aligned}
iG_F(p)|_{p^2 \approx -m^2} =& \frac{1}{p^2 + m^2 - \Pi'(-m^2)(p^2 + m^2) - i\epsilon} \\
=& \frac{1}{p^2 + m^2 - i\epsilon} \frac{1}{1 - \left.\frac{d\Pi}{dp^2}\right|_{p^2 = -m^2}},
\end{aligned}$$

and so $Z^{-1} = 1 - \left.\frac{d\Pi}{dp^2}\right|_{p^2 = -m^2}$.

We now proceed to evaluate $\Pi(p^2)$ to the lowest order in g:

$$i\Pi(p^2) = \tag{14}$$



Using our Feynman rules, we have

$$\begin{aligned}
i\Pi(p^2) =& \frac{1}{2}(ig)^2 \int \frac{d^dq}{(2\pi)^d} G_F^{(0)}(q) G_F^{(0)}(q+p) \\
=& \frac{g^2}{2} \int \frac{d^dq}{(2\pi)^d} \frac{1}{(q^2 + m_0^2 - i\epsilon)((q+p)^2 + m_0^2 - i\epsilon)}.
\end{aligned}$$

We now evaluate the integral explicitly, using a series of tricks. Firstly, we note the identity, due to Feynman:

$$\frac{1}{a_1 \cdots a_n} = \int_0^1 dx_1 \cdots dx_n \, \delta(x_1 + \cdots + x_n - 1)(n-1)!(x_1 a_1 + \cdots + x_n a_n)^{-n}. \tag{15}$$

Thus, our integrand can be rewritten as

$$\begin{aligned}
\frac{1}{(q^2 + m_0^2 - i\epsilon)((q+p)^2 + m_0^2 - i\epsilon)} =& \int_0^1 dx \, \frac{1}{\left[x((q+p)^2 + m_0^2) + (1-x)(q^2 + m_0^2)\right]^2} \\
=& \int_0^1 dx \frac{1}{\left[(q + xp)^2 + D\right]^2}
\end{aligned}$$

with $D = m_0^2 + x(1-x)p^2$. Next, we perform a Wick rotation. In Lorentzian signature, the integral is not convenient to evaluate, because of the poles near the integration paths. Thus, we rotate the q_0 contour to the imaginary axis along the direction shown in figure 1. So, we let $q_0 = iq_d$, and therefore $q^2 = q_1^2 + \ldots + q_d^2 = q_E^2$, and $\int d^dq = i \int d^dq_E$.

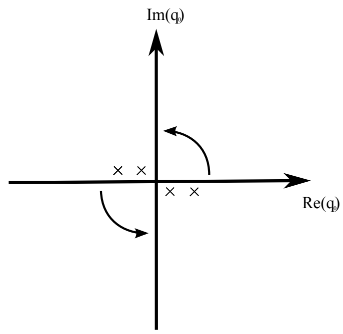


Figure 1: Illustration of the Wick rotation of the variable q_0.

Combining our two results, we have the self-energy in the form

$$i\Pi(p^2) = \frac{ig^2}{2}\int_0^1 dx \int \frac{d^d q_E}{(2\pi)^d}\frac{1}{(q_E^2 + D)^2}. \tag{16}$$

We now proceed to the final evaluation. We observe that (16) is convergent only for $d < 4$. We may evaluate it for $d < 4$, then analytically continue its value for $d \geq 4$, treating d as a complex variable. We make use of the general formula

$$\int \frac{d^d q_E}{(2\pi)^d}\frac{(q_E)^a}{(q_E + D)^b} = \frac{\Gamma(b - a - \frac{d}{2})\Gamma(a + \frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(b)\Gamma(\frac{d}{2})} D^{-(b-a-\frac{d}{2})}. \tag{17}$$

In this case, we have $b = 2$, $a = 0$. And so,

$$\Pi(p^2) = \frac{g^2}{2}\frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\int_0^1 dx \frac{1}{(m_0^2 + x(1-x)p^2)^{2-\frac{d}{2}}}. \tag{18}$$

Example 1: d=3

$$\Pi(p^2) = \frac{g^2}{16\pi}\int_0^1 dx \frac{1}{(m_0^2 + x(1-x)p^2)^{\frac{1}{2}}}. \tag{19}$$

First, we consider the physical mass to $O(g^2)$. We wish to find the solution to (13). Note that the self-energy is of order g^2, and so the solution is given to $O(g^2)$ by

$$\begin{aligned} m^2 =& m_0^2 - \Pi(-m_0^2) \\ =& m_0^2 - \frac{g^2}{16\pi}\frac{1}{m_0}\int_0^1 dx \frac{1}{(1 - x(1-x))^{\frac{1}{2}}} \\ =& m_0^2 - \frac{g^2}{16\pi m_0}\log 3. \end{aligned}$$

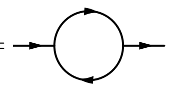
The second term is the mass renormalization due to the interaction, to lowest order. Now, we consider the field renormalization, $Z^{-1} = 1 - \frac{d\Pi}{dp^2}\Big|_{p^2=-m^2}$, and note that again, as $\Pi(p^2)$ is of order g^2, to lowest order we have

$$\begin{aligned} Z^{-1} =& 1 - \frac{d\Pi}{dp^2}\Big|_{p^2=-m_0^2} \\ =& 1 - \frac{g^2}{16\pi}(-\frac{1}{2})\frac{1}{m_0^3}\int_0^1 dx \frac{x(1-x)}{(1 - x(1-x))^{\frac{3}{2}}} \\ =& \frac{1}{1 + \frac{0.23 g^2}{32\pi m_0^3}} < 1, \end{aligned}$$

where the integral in dx has been evaluated explicitly. Finally, we note that for $-p^2 \geq 4m_0^2$, we have that $m_0^2 + x(1-x)p^2$ is smaller than zero for a range of x between 0 and 1. Therefore, $\Pi(p^2)$ becomes complex. It is convenient to consider

$$\Pi(s) = \frac{g^2}{16\pi}\int_0^1 dx\, \frac{1}{(m_0^2 - x(1-x)s)^{\frac{1}{2}}} \tag{20}$$

as a function of a complex variable s, with $\Pi(p^2) = \Pi(s = -p^2 + i\epsilon)$. $\Pi(s)$ has a branch point at $s = 4m_0^2$. This is precisely the multiple-particle cut predicted in the general formalism last lecture. Note that $m^2 = m_0^2 + O(g^2)$. We can now understand the physical interpretation of this result:

$i\Pi(p^2) =$  (21)

to lowest order, and when $-p^2 > 4m_0^2$, both the intermediate particles can simultaneously go on-shell. $\mathfrak{Im}(iG_F) = \pi\sigma(-p^2)$ becomes non-zero and $\mathfrak{Im}(-p^2)$ is just the Feynman diagram evaluated with both the intermediate particles on shell, giving a factor of $\delta(p^2 + m^2)$ for each propagator.